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COMPLETE REDUCIBILITY AND SEPARABILITY

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ABSTRACT. Let G be a reductive linear algebraic group over an algebraically closed field of characteristic $p > 0$. A subgroup of G is said to be separable in G if its global and infinitesimal centralizers have the same dimension. We study the interaction between the notion of separability and Serre's concept of G-complete reducibility for subgroups of G. A separability hypothesis appears in many general theorems concerning G-complete reducibility. We demonstrate that some of these results fail without this hypothesis. On the other hand, we prove that if G is a connected reductive group and p is very good for G, then any subgroup of G is separable; we deduce that under these hypotheses on G, a subgroup H of G is G-completely reducible provided Lie G is semisimple as an H-module.

Recently, Guralnick has proved that if H is a reductive subgroup of G and C is a conjugacy class of G, then $C \cap H$ is a finite union of H-conjugacy classes. For generic p — when certain extra hypotheses hold, including separability — this follows from a well-known tangent space argument due to Richardson, but in general, it rests on Lusztig's deep result that a connected reductive group has only finitely many unipotent conjugacy classes. We show that the analogue of Guralnick's result is false if one considers conjugacy classes of n-tuples of elements from H for $n > 1$.

CONTENTS

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1. Introduction

Let G be a reductive algebraic group over an algebraically closed field k. A subgroup H of G is said to be G-completely reducible (G-cr) if whenever H is contained in a parabolic subgroup P of G, H is contained in some Levi subgroup of P. (If G is non-connected, then we have to modify this definition slightly; see Subsection 2.2.) In the special case $G = \mathrm{GL}_n(k)$, H is G-completely reducible if and only if H acts completely reducibly on k^n. Thus G-complete reducibility generalizes the notion of semisimplicity from representation theory, [30, 31, 32].

Much effort has been made to study G-completely reducible subgroups of reductive groups. There are applications to finite groups of Lie type [17]. Ideas from the theory of G-complete reducibility play an important part in the study by Liebeck and Seitz [18] of the reductive subgroups of the exceptional simple groups.

In characteristic zero a subgroup H of G is G-completely reducible if and only if H is reductive (cf. [3, Lem. 2.6]). In this case, it is trivial to show that the class of G-completely reducible subgroups of G is closed under certain natural constructions: for instance, if H, N are G-completely reducible subgroups of G such that H normalizes N, then H and N are reductive, so HN is reductive, and thus HN is G-completely reducible. The analogous result need not hold in positive characteristic [4, Examples 5.1 and 5.3]. Nonetheless, results concerning G-completely reducible subgroups which hold in characteristic zero tend to hold "generically" in positive characteristic, that is, if the characteristic is large enough, or under certain extra natural restrictions on the subgroups considered. We give some other examples below: see Theorem 1.7 and Theorem 4.4.

The purpose of this paper is to explore the bounds of these generic results, proving that they hold without the extra hypotheses or finding counterexamples when the extra hypotheses are removed. Characteristic 2 is a particularly fertile ground for counterexamples. To prove positive results, we use geometric techniques from our earlier works [3] and [4]. Especially important is the property of *separability* (see [3, Def. 3.27]): the hypothesis that certain subgroups are separable is required for several results involving G-complete reducibility, e.g., [3, Thm. 3.35, Thm. 3.46].

Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G.

Definition 1.1. A subgroup H of G is said to be *separable in* G if $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$, that is, if the scheme-theoretic centralizer of H in G is smooth.

Here $C_G(H)$ and $\mathfrak{c}_{\mathfrak{g}}(H)$ denote the (set-theoretic) centralizer of H in G and in $\mathfrak{g}$, respectively. Note that we always have $\operatorname{Lie} C_G(H) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$.

The motivation for this terminology is as follows. Given $n \in \mathbb{N}$, we let G act on G^n by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

Suppose H is the algebraic subgroup of G generated by elements $g_1, \ldots, g_n \in G$. Then we say H is *(topologically) finitely generated* by $g_1, \ldots, g_n$. The orbit map $G \to G \cdot (g_1, \ldots, g_n)$, $g \mapsto g \cdot (g_1, g_2, \ldots, g_n)$ is a separable morphism of varieties if and only if H is separable in G (see [3, Sec. 3.5]).

There is an analogous notion of separability for subalgebras of $\mathfrak{g}$ — see Definition 2.10 — which similarly is related to the separability of orbit maps $G \to G \cdot (x_1, \ldots, x_n)$, where $x_i \in \mathfrak{g}$ and G acts diagonally on $\mathfrak{g}^n$ by simultaneous adjoint action.

Our first main result concerning this notion of separability is as follows.

Theorem 1.2. *Let G be connected reductive and suppose that* $\mathrm{Char}\, k$ *is very good for G. Then any subgroup of G is separable in G and any subalgebra of $\mathfrak{g}$ is separable in $\mathfrak{g}$.*

The proof (see Section 3) depends on Theorem 1.4 below and the existence of reductive pairs under the given characteristic restrictions, where for a reductive subgroup H of a reductive group G, we say that (G, H) is a *reductive pair* if the Lie algebra $\mathfrak{h} = \mathrm{Lie}\, H$ is an H-module direct summand of $\mathfrak{g}$ (of course, this is automatically satisfied in characteristic zero). Special cases of Theorem 1.2 are known due to work by Liebeck–Seitz [18, Thm. 3] and Lawther–Testerman [16, Thm. 2].

A prototype of the results we consider is the following fundamental result of P. Slodowy [34, Thm. 1], which is obtained by applying a standard tangent space argument due to R.W. Richardson (cf. [25, Sec. 3] and [26, Lem. 3.1]). Theorem 1.3 has many applications; see [3], [34], [39, Sec. 3], for example. Some of the results that follow from Theorem 1.3 are still valid even when the hypotheses of the theorem are not met, but one often has to work much harder to obtain them (cf. [22, Sec. 1]).

Theorem 1.3. *Let H be a reductive subgroup of a reductive group G. Let $n \in \mathbb{N}$, let $(h_1, \dots, h_n) \in H^n$ and let K be the algebraic subgroup of H generated by $h_1, \dots, h_n$. Suppose that (G, H) is a reductive pair and that K is separable in G. Then*

(a) *for all $(g_1, \dots, g_n) \in G \cdot (h_1, \dots, h_n) \cap H^n$, the H-orbit map $H \to H \cdot (g_1, \dots, g_n)$ is separable. In particular, K is a separable subgroup of H;*
(b) *the intersection $G \cdot (h_1, \dots, h_n) \cap H^n$ is a finite union of H-conjugacy classes;*
(c) *each H-conjugacy class in $G \cdot (h_1, \dots, h_n) \cap H^n$ is closed in $G \cdot (h_1, \dots, h_n) \cap H^n$.*

The following consequence [3, Thm. 3.35] of Theorem 1.3 gives an application to G-complete reducibility. The second assertion follows from Theorem 1.3(c) and the geometric characterization of G-complete reducibility given in [3] (see Theorem 2.9 below); the first does not appear explicitly in [3, Thm. 3.35], but it follows immediately from the proof together with Theorem 1.3(a).

Theorem 1.4. *Suppose that (G, H) is a reductive pair. Let K be a subgroup of H such that K is a separable subgroup of G. Then K is separable in H. Moreover, if K is G-completely reducible, then it is also H-completely reducible.*

Theorem 1.3(b) can be used to give a simple proof that a connected reductive group G has only finitely many conjugacy classes of unipotent elements in generic characteristic: one takes an embedding of G in some $\mathrm{GL}(V)$ such that $(\mathrm{GL}(V), G)$ is a reductive pair, then applies Theorem 1.3(b) (taking $n = 1$) to deduce the result for G from the result for $\mathrm{GL}(V)$. In small positive characteristic there need not exist any such reductive pair $(\mathrm{GL}(V), G)$ (see Subsection 2.3). Nonetheless Lusztig proved without any restriction on the characteristic, using some very deep mathematics, that G has only finitely many unipotent conjugacy classes [20]. Guralnick recently extended this result to non-connected reductive groups [10, Thm. 3.3]; he then used it to prove that Theorem 1.3(b) still holds for an arbitrary reductive subgroup H of G if $n = 1$ [10, Thm. 1.2], even though Richardson's proof no longer goes through.

In view of this, it is natural to ask whether Theorem 1.3(b) holds in more generality. In Example 7.15, we prove the following result, which shows that Guralnick's conjugacy result [10, Thm. 1.2] does not extend to conjugacy classes of n-tuples for $n > 1$.

Theorem 1.5. *There exists a reductive group G, a reductive subgroup M of G and a pair $(m_1, m_2) \in M^2$ such that (G, M) is a reductive pair but $G \cdot (m_1, m_2) \cap M^2$ is* **not** *a finite union of M-conjugacy classes.*

Thus Theorem 1.3(b) is false in general without the separability hypothesis. We give an example (Proposition 7.17) showing that the second assertion of Theorem 1.4 also fails without the separability hypothesis. This implies that Theorem 1.3(c) fails as well without the separability hypothesis (see Remark 7.18).

The paper is split into several sections, as we now outline. In Section 2 we introduce the notation and preliminary results needed for the rest of the exposition. In particular, we recall the formalism of R-parabolic subgroups from [3], which allows us to consider reductive groups which are not connected; this is very important for many of our results.

In Section 3 we discuss the question of separability in connection with reductive pairs. In Proposition 3.7 we give a construction for certain reductive pairs, where the reductive subgroup of G is of the form $C_G(S)$; here S is a reductive group acting suitably on G.

In Section 4 we investigate the connection between the G-complete reducibility of a subgroup H and the semisimplicity of the adjoint representation of H on $\mathfrak{g}$. The following result is the basis of our discussion [3, Thm. 3.46].

Theorem 1.6. *Let H be a separable subgroup of G. If $\mathfrak{g}$ is semisimple as an H-module, then H is G-completely reducible.*

One way of removing the separability hypothesis from Theorem 1.6 is to combine Theorems 1.2 and 1.6. This immediately gives the following result.

Theorem 1.7. *Let G be connected reductive and suppose that* $\operatorname{Char} k$ *is very good for G. Let H be a subgroup of G such that $\mathfrak{g}$ is a semisimple H-module. Then H is G-completely reducible.*

The assumption in Theorem 1.7 that $\operatorname{Char} k$ is very good for G is rather restrictive. In Section 4 we discuss to what extent we can remove the separability hypothesis from Theorem 1.6 with a weaker assumption on $\operatorname{Char} k$; see Theorem 4.1, Theorem 4.4 — which is our second main result — and Corollary 4.5.

In Section 5 we extend two general results of Richardson concerning orbits of reductive groups on affine varieties [26, Thm. A, Thm. C]; see Theorem 5.4. We apply these extensions in turn to questions of G-complete reducibility; in particular, we discuss some results which examine the relationship between G-complete reducibility and H-complete reducibility of a subgroup K of H, where H is a subgroup of G of the form $H = C_G(S)$, with S a reductive group acting suitably on G; see Proposition 5.7. The case when the group S considered above is a subgroup of G is discussed in Section 6.

In Section 7, we provide an important collection of closely related constructions. They give counterexamples to several of our results, including Theorems 1.3 and 1.4, under weakened hypotheses. In particular, here we also prove Theorem 1.5.

2. Preliminaries

2.1. **Notation.** Throughout, we work over an algebraically closed field k; we let k^* denote the multiplicative group of k. All algebraic groups are assumed to be linear. By a subgroup of an algebraic group we mean a closed subgroup, and by a homomorphism of algebraic groups we mean a homomorphism of abstract groups that is also a morphism of algebraic varieties. Let H be a linear algebraic group. We denote by $\overline{\langle S\rangle}$ the algebraic subgroup of H generated by a subset S. We let DH denote the derived group $[H,H]$, $Z(H)$ the centre of H, and H^0 the connected component of H that contains 1. If S is a subset of H, then $C_H(S)$ is the centralizer of S in H and $N_H(S)$ is the normalizer of S in H. In general we use an upper-case roman letter, G, H, K, etc., to denote an algebraic group and the corresponding lower-case gothic letter, $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$, etc., to denote its Lie algebra. If $\mathfrak{h}$ is a Lie algebra and S is a subset of $\mathfrak{h}$, then $\mathfrak{c}_{\mathfrak{h}}(S)$ is the centralizer of S in $\mathfrak{h}$. We denote the centre of $\mathfrak{h}$ by $\mathfrak{z}(\mathfrak{h})$.

Let $\mathrm{Ad}: H \to \mathrm{GL}(\mathfrak{h})$ denote the adjoint representation; then we let H_{ad} denote the image of H under this map and $\mathfrak{h}_{\mathrm{ad}}$ denote $\mathrm{Lie}\, H_{\mathrm{ad}}$. Note that $(H_{\mathrm{ad}})^0$ is the adjoint form of $D(H^0)$ [7, V.24.1].

For the set of cocharacters (one-parameter subgroups) of H we write $Y(H)$; the elements of $Y(H)$ are the homomorphisms from k^* to H.

The *unipotent radical* of H is denoted $R_u(H)$; it is the maximal connected normal unipotent subgroup of H. The algebraic group H is called *reductive* if $R_u(H) = \{1\}$; note that we do not insist that a reductive group is connected. In particular, H is reductive if it is simple as an algebraic group (H is said to be *simple* if H is connected and all proper normal subgroups of H are finite). If N is a normal subgroup of H, then H is reductive if and only if N and H/N are. The algebraic group H is called *linearly reductive* if all rational representations of H are semisimple.

If H acts on the affine variety X, then we denote by X^H the fixed point subvariety of X: that is, $X^H = \{x \in X \mid h \cdot x = x \ \forall h \in H\}$. If S is a subset of X, then we denote the pointwise stabilizer of S in H by $C_H(S)$; we write $C_H(x)$ instead of $C_H(\{x\})$ for x in X. If $X = K$ is an algebraic group and H acts on K by automorphisms, then we write $C_K(H)$ instead of K^H. Then we also have an induced linear action of H on $\mathfrak{k} = \mathrm{Lie}\, K$; we write $\mathfrak{c}_{\mathfrak{k}}(H)$ instead of $\mathfrak{k}^H$.

Throughout the paper G denotes a reductive algebraic group, possibly non-connected, with Lie algebra $\mathfrak{g}$. A subgroup of G normalized by some maximal torus of G is called a *regular* subgroup of G (connected reductive regular subgroups of connected reductive groups are often also referred to as *subsystem subgroups*, e.g., see [19]).

Fix a maximal torus T of G. We write $X(T)$ for the character group of T. Let $\Psi = \Psi(G,T) \subseteq X(T)$ denote the set of roots of G with respect to T. We write $\mathfrak{t} = \mathrm{Lie}\, T$ for the Lie algebra of T. If $\alpha \in \Psi$, then U_α denotes the corresponding root subgroup of G and $\mathfrak{u}_\alpha$ denotes the root space $\mathrm{Lie}\, U_\alpha$ of $\mathfrak{g}$. Thus the root space decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{u}_\alpha.$$

We denote by G_γ the simple rank 1 subgroup $\langle U_\gamma \cup U_{-\gamma}\rangle$ of G and by $\mathfrak{g}_\gamma$ the Lie algebra of G_γ. Fix a Borel subgroup B of G containing T and let $\Sigma = \Sigma(G,T)$ be the set of simple roots of Ψ defined by B. Then $\Psi^+ = \Psi(B,T)$ is the set of positive roots of G. For $\beta \in \Psi^+$ write $\beta = \sum_{\alpha\in\Sigma} c_{\alpha\beta}\alpha$ with $c_{\alpha\beta} \in \mathbb{N}_0$. A prime p is said to be *good* for G if it does not divide any non-zero $c_{\alpha\beta}$, and *bad* otherwise. A prime p is good for G if and only if it is good for every simple factor of G^0 [36]; the bad primes for the simple groups are 2 for all groups except type A_n, 3 for the exceptional groups and 5 for type E_8. A prime p is said to be *very good* for G if p is good for G and p does not divide $n+1$ for any simple component of G of type A_n. If G is simple and $\operatorname{Char} k$ is very good for G, then the Lie algebra $\mathfrak{g}$ is simple [37].

Remark 2.1. Separability of subgroups of G and of subalgebras of $\mathfrak{g}$ (see Definition 2.10) is automatic in characteristic zero (cf. [14, Thm. 13.4]). Likewise, the notion of G-complete reducibility is not interesting in characteristic zero, as a subgroup of G is G-completely reducible if and only if it is reductive (cf. [3, Lem. 2.6]); most of our results and proofs become trivial in characteristic zero. In the remainder of the paper, p denotes the characteristic $\operatorname{Char} k$ of k in case $\operatorname{Char} k > 0$.

Let $\gamma \in \Psi$. We denote by $\gamma^\vee \in Y(G)$ the corresponding coroot. Then $\gamma^\vee$ is a homomorphism from k^* to G_γ. If $\alpha, \beta \in \Sigma$, then we have $s_\alpha \cdot \beta = \beta - \langle \beta, \alpha^\vee\rangle\alpha$ [35, Lem. 7.1.8], where s_α is the reflection corresponding to α in the Weyl group of G.

In Section 7 we need the following well-known result (which is implicit for instance in [15, 6.5]). For convenience we include a proof.

Lemma 2.2. *Assume that G is connected. If the derived group DG of G is simply connected, then the same holds for any Levi subgroup of any parabolic subgroup of G.*

Proof. We use the following characterization of simply connectedness of DG: Let T be a maximal torus of G and let $\alpha_1, \dots, \alpha_n \in X(T)$ be the choice of simple roots corresponding to some Borel subgroup B containing T. Then DG is simply connected if and only if there exist characters $\chi_1, \dots, \chi_n \in X(T)$ such that $\langle \chi_i, \alpha_j^\vee\rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$. For G semisimple this is clear. The general case follows from the fact that the integer $\langle \chi, \alpha^\vee\rangle$ only depends on the restriction of χ to $T \cap DG$ and the fact that any character of $T \cap DG$ can be lifted to a character of T ([7, Prop. III.8.2(c)]).

Let P be a parabolic subgroup of G and let L be a Levi subgroup of P. We may assume that P contains B and L contains T. The result now follows immediately from the following well-known description of L [7, Prop. IV.14.18]: the simple roots of L can be chosen from the set $\{\alpha_1, \dots, \alpha_n\}$. $\square$

The next result allows us in positive characteristic to replace an algebraic group S acting on G by automorphisms with a finite subgroup of S. It is a slight strengthening of [5, Thm. 7], since it asserts the existence of a *countable* locally finite dense subgroup. We use this lemma in the proof of Proposition 3.7 and also in several places in Section 5.

Lemma 2.3. *Assume that* $\operatorname{Char} k > 0$. *Let H be a linear algebraic group over k. Then there exists an ascending sequence $H_1 \subseteq H_2 \subseteq \cdots$ of finite subgroups of H whose union is dense in H.*

Proof. We proceed by induction on $\dim H$. If H is reductive, then the result follows from [22, Sec. 3]. Otherwise $Z := Z(R_u(H))^0$ is a connected unipotent normal subgroup of H of dimension ≥ 1. By [7, III.10.6(2)], Z contains a subgroup isomorphic to the additive group $\mathbb{G}_a$. Let C be the subgroup of Z generated by the subgroups of Z that are isomorphic to $\mathbb{G}_a$. Then C is the additive group of a non-zero finite-dimensional vector space over k, by [11, Thm. 5.4]. Furthermore, C is normal in G. Clearly, C has an ascending sequence $C_1 \subseteq C_2 \subseteq \cdots$ of finite subgroups whose union is dense in C. By the induction hypothesis, $M := H/C$ also has an ascending sequence $M_1 \subseteq M_2 \subseteq \cdots$ of finite subgroups whose union is dense in M. Let $\pi : H \to M$ be the canonical projection. For each $i \geq 1$ let H_i be a finitely generated subgroup of H such that $\pi(H_i) = M_i$. Without loss of generality we may assume that the H_i form an ascending sequence of subgroups and that H_i contains C_i. Since H_i is finitely generated and $H_i \cap C$ is of finite index in H_i, we have that $H_i \cap C$ is finitely generated, by [28, Thm. 11.54]. Since C is a vector space, this means that $H_i \cap C$ is finite. But then H_i is finite. Now let H' be the closure of the union of the H_i. Then H' is a closed subgroup of H containing C. Its image $\pi(H')$ is a closed subgroup of M containing the M_i and is therefore equal to M. Consequently, $H' = H$. □

2.2. **G-complete reducibility.** In [3, Sec. 6], Serre's original notion of G-complete reducibility is extended to include the case when G is reductive but not necessarily connected (so that G^0 is a connected reductive group). The crucial ingredient of this extension is the introduction of so-called *Richardson-parabolic subgroups* (*R-parabolic subgroups*) of G. We briefly recall the main definitions here; for more details on this formalism, see [3, Sec. 6].

For a cocharacter $\lambda \in Y(G)$, the *R-parabolic subgroup* corresponding to λ is defined by $P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$. Then P_λ admits a Levi decomposition $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$, where $L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\} = C_G(\lambda(k^*))$. We call L_λ an *R-Levi subgroup* of P_λ. For an R-parabolic subgroup P of G, the different R-Levi subgroups of P correspond in this way to different choices of $\lambda \in Y(G)$ such that $P = P_\lambda$; moreover, the R-Levi subgroups of P are all conjugate under the action of $R_u(P)$. An R-parabolic subgroup P is a parabolic subgroup in the sense that G/P is a complete variety; the converse is true when G is connected, but not in general ([22, Rem. 5.3]). The map $c_\lambda : P_\lambda \to L_\lambda$ given by $c_\lambda(g) = \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1}$ is a surjective homomorphism of algebraic groups with kernel $R_u(P_\lambda)$; it coincides with the usual projection $P_\lambda \to L_\lambda$.

Remark 2.4. For a subgroup H of G, there is a natural inclusion $Y(H) \subseteq Y(G)$. If $\lambda \in Y(H)$, and H is reductive, we can therefore associate to λ an R-parabolic subgroup of H as well as an R-parabolic subgroup of G. To avoid confusion, we reserve the notation P_λ for R-parabolic subgroups of G, and distinguish the R-parabolic subgroups of H by writing $P_\lambda(H)$ for $\lambda \in Y(H)$. The notation $L_\lambda(H)$ has the obvious meaning. Note that $P_\lambda(H) = P_\lambda \cap H$, $L_\lambda(H) = L_\lambda \cap H$ and $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$ for $\lambda \in Y(H)$.

Remark 2.5. If $\lambda \in Y(G)$ and $P_\lambda^0 = G^0$, then P_λ is an R-Levi subgroup of itself: for $R_u(P_\lambda)$, being connected, is contained in G^0, so $R_u(P_\lambda) = R_u(P_\lambda) \cap G^0 = R_u(P_\lambda(G^0)) = R_u(P_\lambda \cap G^0) = R_u(G^0) = \{1\}$.

Definition 2.6. Suppose H is a subgroup of G. We say H is *G-completely reducible* (G-cr for short) if whenever H is contained in an R-parabolic subgroup P of G, then there exists an R-Levi subgroup L of P with $H \subseteq L$.

Remark 2.7. If H is a G-completely reducible subgroup of G, then H is reductive (cf. [3, Sec. 2.5 and Thm. 3.1]).

Since all parabolic subgroups (respectively all Levi subgroups of parabolic subgroups) of a connected reductive group are R-parabolic subgroups (respectively R-Levi subgroups of R-parabolic subgroups), Definition 2.6 coincides with Serre's original definition for connected groups [32]. Sometimes we come across subgroups of G which are not contained in any proper R-parabolic subgroup of G; these subgroups are trivially G-completely reducible. Following Serre again, we call these subgroups *G-irreducible* (G-ir).

Remark 2.8. Since R-Levi subgroups of R-parabolic subgroups play an important rôle in many of our proofs, for brevity we sometimes abuse language and refer to an *R-Levi subgroup of* G; by this we mean an R-Levi subgroup of some R-parabolic subgroup of G. Similarly, when G is connected, we may refer to *a Levi subgroup of* G; this means a Levi subgroup of some parabolic subgroup of G.

A key result is the following [3, Cor. 3.7], which gives a geometric criterion for G-complete reducibility.

Theorem 2.9. *Let $g_1, \ldots, g_n \in G$ and let $H = \overline{\langle \{g_1, \ldots, g_n\} \rangle}$. Then H is G-completely reducible if and only if the conjugacy class $G \cdot (g_1, \ldots, g_n)$ is closed in G^n.*

We frequently require results from [3, Sec. 6.3] for non-connected G, though we usually simply cite the relevant result in [3] for connected G.

2.3. **Separability.** In Section 3 we require the following analogue of Definition 1.1 for subalgebras of $\mathfrak{g}$. Recall that $C_G(\mathfrak{h}) = \{g \in G \mid \operatorname{Ad} g(x) = x \text{ for all } x \in \mathfrak{h}\}$.

Definition 2.10. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *separable in* $\mathfrak{g}$ if $\operatorname{Lie} C_G(\mathfrak{h}) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.

The above definition has the same motivation as in the group case. Given $n \in \mathbb{N}$, we let G act on $\mathfrak{g}^n$ by diagonal adjoint action. Suppose $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ generated by elements $x_1, \ldots, x_n \in \mathfrak{g}$. Then the orbit map $G \to G \cdot (x_1, \ldots, x_n)$, $g \mapsto g \cdot (x_1, \ldots, x_n)$ is a separable morphism of varieties if and only if $\mathfrak{h}$ is separable in $\mathfrak{g}$ (see [7, II.6.7]).

As in the group case ([34, Thm. 1]), it is straightforward to generalize Richardson's tangent space arguments [25, Sec. 3] and [26, Lem. 3.1] to the action of G on $\mathfrak{g}^n$. We then obtain the following analogue of Theorem 1.3.

Theorem 2.11. *Let H be a reductive subgroup of G. Let $n \in \mathbb{N}$, let $(x_1, \ldots, x_n) \in \mathfrak{h}^n$ and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{h}$ generated by $x_1, \ldots, x_n$. Suppose that (G, H) is a reductive pair and that $\mathfrak{k}$ is separable in $\mathfrak{g}$. Then*

(a) *all H-orbit maps in $G \cdot (x_1, \ldots, x_n) \cap \mathfrak{h}^n$ are separable. In particular, $\mathfrak{k}$ is separable in $\mathfrak{h}$;*
(b) *the intersection $G \cdot (x_1, \ldots, x_n) \cap \mathfrak{h}^n$ is a finite union of H-conjugacy classes;*
(c) *each H-conjugacy class in $G \cdot (x_1, \ldots, x_n) \cap \mathfrak{h}^n$ is closed in $G \cdot (x_1, \ldots, x_n) \cap \mathfrak{h}^n$.*

We immediately obtain the following analogue of the first assertion of Theorem 1.4.

Corollary 2.12. *Suppose that (G, H) is a reductive pair. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{h}$ such that $\mathfrak{k}$ is separable in $\mathfrak{g}$. Then $\mathfrak{k}$ is separable in $\mathfrak{h}$.*

Because every subgroup of $\mathrm{GL}(V)$ is separable in $\mathrm{GL}(V)$ and every subalgebra of $\mathfrak{gl}(V)$ is separable in $\mathfrak{gl}(V)$ (e.g., see [3, Ex. 3.28]), Theorem 1.4 and Corollary 2.12 imply the following.

Corollary 2.13. *If $(\mathrm{GL}(V), G)$ is a reductive pair, then every subgroup of G is separable in G and every subalgebra of $\mathfrak{g}$ is separable in $\mathfrak{g}$.*

We deduce from this that not every G can be embedded in some $\mathrm{GL}(V)$ in such a way that $(\mathrm{GL}(V), G)$ is a reductive pair: this applies, for instance, to $G = \mathrm{SL}_2(k)$ when $p = 2$, because $H = G$ is not a separable subgroup of G. However, if G is of a given Dynkin type, then generically — that is, for almost all values of p — the conclusion of Corollary 2.13 holds; for example, if G is an exceptional simple group of adjoint type and p is good for G, then $(\mathrm{GL}(\mathfrak{g}), G)$ is a reductive pair (cf. Example 4.7).

The final result of this section shows that a non-separable G-cr subgroup K of G is, up to isogeny, a separable subgroup of a regular subgroup of G. Given a reductive group M, we let $\pi_M : M \to M_{\mathrm{ad}}$ denote the natural morphism.

Proposition 2.14. *Let K be a G-completely reducible subgroup of G. Then there exists a reductive subgroup M of G containing a maximal torus of G such that $K \subseteq M$, K is M-irreducible and $\pi_M(K)$ is separable in M_{ad}.*

Proof. Since K is G-cr, we may assume by [3, Cor. 3.5] that K is G-ir after replacing G by an R-Levi subgroup of G that is minimal with respect to containing K. By [3, Cor. 2.7(i)], K is M-ir in any reductive subgroup M of G containing K. If $\pi_G(K)$ is separable in G_{ad}, then we can take $M = G$, so suppose not. By [3, Prop. 3.39], there exists a reductive subgroup M' of G_{ad} containing a maximal torus of G_{ad} such that $\pi_G(K) \subseteq M'$ and M' is not separable in G_{ad}. As $(G_{\mathrm{ad}})^0$ is of adjoint type, its Lie algebra has trivial centre, so any overgroup of $(G_{\mathrm{ad}})^0$ is separable in G_{ad}. This forces M' to be of strictly smaller dimension than G_{ad}. Let $M = \pi_G^{-1}(M')$, a subgroup of G which is of strictly smaller dimension than G and contains a maximal torus of G. Since M is an overgroup of the G-ir group K, M is reductive. The result now follows by induction on $\dim G$. $\square$

Proposition 2.14 is false if we do not assume that K is G-completely reducible: see Example 7.20 below.

3. Reductive pairs and separability

The notion of separability is central to many of the results in this paper. Theorems 1.3 and 1.4 both illustrate the importance of reductive pairs in this context. In this section we elaborate on this theme. For examples and constructions of reductive pairs, we refer to [34, Sec. I.3] and [3, Sec. 3.5].

Recall that an isogeny is an epimorphism with finite kernel and that it is called *separable* if its differential is an isomorphism.

Lemma 3.1. *Let $\varphi : G \to G'$ be a separable isogeny of reductive groups, let H be a subgroup of G and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Then H is separable in G if and only if $\varphi(H)$ is separable in G' and $\mathfrak{k}$ is separable in $\mathfrak{g}$ if and only if $d\varphi(\mathfrak{k})$ is separable in $\mathfrak{g}'$.*

Proof. Since the differential $d\varphi$ of φ is an isomorphism, it is clear that $d\varphi(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{k})) = \mathfrak{c}_{\mathfrak{g}'}(d\varphi(\mathfrak{k}))$. Furthermore, we have

$$\mathrm{Ad}(\varphi(g)) \circ d\varphi = d\varphi \circ \mathrm{Ad}(g) \quad \text{for all } g \in G, \tag{3.2}$$

from which it follows that $d\varphi(\mathfrak{c}_{\mathfrak{g}}(H)) = \mathfrak{c}_{\mathfrak{g}'}(\varphi(H))$.

So to prove the first statement it suffices to show that $\varphi(C_G(H))^0 = C_{G'}(\varphi(H))^0$. For this, in turn, it suffices to show that $\varphi^{-1}(C_{G'}(\varphi(H)))^0 \subseteq C_G(H)$. If $y \in H$, then the image of $\varphi^{-1}(C_{G'}(\varphi(H)))^0$ under the morphism $x \mapsto xyx^{-1}y^{-1}$ is an irreducible subset of the finite set $\ker\varphi$ which contains 1, so it must equal $\{1\}$.

To prove the second statement, it suffices to show that $\varphi(C_G(\mathfrak{k})) = C_{G'}(d\varphi(\mathfrak{k}))$. This follows easily from (3.2). $\square$

Lemma 3.3. *Let G be connected and suppose that p is very good for G. Then there exists a separable isogeny $S \times H \to G$, where S is a torus and H is a product of simply connected simple groups.*

Proof. Let $G_1, \dots, G_r$ be the simple factors of DG and let $\widetilde{G}_i$ be the simply connected cover of G_i for each i. Then $\mathrm{Lie}\,\widetilde{G}_i$ is simple for each i, by our hypothesis on p. Set $S = Z(G)^0$ and $H = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$. It is easily checked that the multiplication map $S \times H \to G$ is a separable isogeny. $\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.3, we may assume that $G = S \times H_1 \times \cdots \times H_r$, where S is a torus and each H_i is a simply connected simple group. Put $H_0 = S$ and let $\pi_i : G \to H_i$ be the projection. For every subgroup K of G we have $C_G(K) = \prod_{i=0}^r C_{H_i}(\pi_i(K))$ and for every subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ we have $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{m}) = \bigoplus_{i=0}^r \mathfrak{c}_{\mathfrak{h}_i}(d\pi_i(\mathfrak{m}))$. Since S is abelian, we may now assume that G is a simply connected simple group.

It now suffices to prove that there exists a simple group G', a separable isogeny $\eta\colon G \to G'$ and an embedding of G' in some $\mathrm{GL}(V')$ such that $(\mathrm{GL}(V'), G')$ is a reductive pair: for then every subgroup of G is separable in G and every subalgebra of $\mathfrak{g}$ is separable in $\mathfrak{g}$, by Corollary 2.13 and Lemma 3.1. For every simple group K of the same Dynkin type as G, the natural isogeny $G \to K$ is separable, since p is very good for G. So to complete the proof, it is enough to show that for every Dynkin type, there exists a simple group K of this Dynkin type and an embedding of K in some $\mathrm{GL}(V)$ such that $(\mathrm{GL}(V), K)$ is a reductive pair.

If $K = \mathrm{SO}(V)$ or $K = \mathrm{Sp}(V)$ and $p \neq 2$, then $(\mathrm{GL}(V), K)$ is a reductive pair [25, Lem. 5.1]. This deals with types B_n, C_n and D_n. If $K = \mathrm{SL}(V)$, then, since p is very good for G, it follows that $(\mathrm{GL}(V), K)$ is a reductive pair: the scalar matrices form a K-stable direct complement to $\mathfrak{sl}(V)$ in $\mathfrak{gl}(V)$. This deals with type A_n. If K is an adjoint simple group of exceptional type and p is good for K, then $(\mathrm{GL}(\mathfrak{k}), K)$ is a reductive pair, thanks to [25, §5]. This completes the proof. $\square$

Corollary 3.4. *Let G be connected and suppose that p is very good for G. Let $g_1, \dots, g_n \in G$ and let $x_1, \dots, x_m \in \mathfrak{g}$. Then the orbit maps $G \to G \cdot (g_1, \dots, g_n)$ and $G \to G \cdot (x_1, \dots, x_m)$ are separable.*

Proof. The orbit map $G \to G \cdot (g_1, \dots, g_n)$ is separable if and only if the algebraic subgroup of G generated by $g_1, \dots, g_n$ is separable in G and the orbit map $G \to G\cdot(x_1, \dots, x_m)$ is separable if and only if the subalgebra of $\mathfrak{g}$ generated by $x_1, \dots, x_m$

is separable in $\mathfrak{g}$ (see the comment after Definition 1.1 and Subsection 2.3), so the corollary follows immediately from Theorem 1.2. □

Remarks 3.5. (i). Consider the class of reductive groups G that have the property that each subgroup of G is separable in G and each subalgebra of $\mathfrak{g}$ is separable in $\mathfrak{g}$. Lemma 3.1, the proof of Theorem 1.2 and Corollary 3.8 below show that this class is closed under separable isogenies (in both directions), direct products and centralizers of subgroups S acting on G by automorphisms as in Proposition 3.7. In particular our class contains the "strongly standard" reductive groups of [24, §2.4].

(ii). For G simple of exceptional type and for simple subgroups of G and $p > 7$, Theorem 1.2 is due to case-by-case checks of Liebeck–Seitz [18, Thm. 3] and Lawther–Testerman [16, Thm. 2].

(iii). The restriction on p in Theorem 1.2 is necessary. For instance, for $G = \mathrm{SL}(V)$ with $\dim V = p$, the group G is not separable in itself. Also for G simple of exceptional type and p a bad prime for G, the pair $(\mathrm{GL}(\mathfrak{g}), G)$ need no longer be a reductive pair, so the proof breaks down: for instance, in Proposition 7.11, we provide an example for G of type G_2 and $p = 2$ of a non-separable subgroup of G (cf. Corollary 2.13).

(iv). The requirement in Theorem 1.2 that G be connected is also necessary. For instance, if $G = k^* \rtimes C_2$, where the non-trivial element c of the cyclic group C_2 acts on k^* by $c \cdot a = a^{-1}$, then C_2 is a non-separable subgroup of G.

(v). The case $n = 1$ in Corollary 3.4 is a well-known fundamental result due to P. Slodowy, [33, p. 38].

(vi). Serre has asked whether Theorem 1.2 holds for an arbitrary group subscheme H of G; Theorem 1.2 deals with two special cases: H smooth and H infinitesimal of height one.

We finish the section with some further useful results on separability and reductive pairs.

Lemma 3.6. *Suppose G is connected. Let S be an algebraic group acting faithfully on G by automorphisms. Then the corresponding representation of S on $\mathfrak{g}$ has finite kernel.*

Proof. It is enough to prove that $S = \{1\}$ under the extra hypotheses that S is connected and S acts trivially on $\mathfrak{g}$, so we assume this. Let $s \in S$. Let B, B^- be any pair of opposite Borel subgroups of G. Since $s \cdot B$ is also a Borel subgroup of B, there exists $g \in G$ such that $s \cdot B = gBg^{-1}$. Let $\mathfrak{b} = \mathrm{Lie}\, B$. Then $\mathfrak{b} = s \cdot \mathfrak{b} = \mathrm{Ad}\, g(\mathfrak{b})$, so $g \in B$, by [7, IV.14.1 Cor. 2], so s normalizes B. Similarly s normalizes B^-, so s normalizes the maximal torus $T = B \cap B^-$ of G. Hence s centralizes T, by the rigidity of tori [7, Prop. III.8.10]. Since B and B^- were arbitrary, s centralizes the set of semisimple elements of G, which is dense in G as G is reductive (e.g., see [7, Thm. IV.12.3(2)]). Thus s centralizes G. But the S-action on G is faithful, so $S = \{1\}$ as required. □

Proposition 3.7. *Let S be an algebraic group acting on G by automorphisms. Suppose that S acts semisimply on $\mathfrak{g}$ and $\mathrm{Lie}\, C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$. Then*

(a) *$C_G(S)$ is G-completely reducible and $(G, C_G(S))$ is a reductive pair.*
(b) *If S is a subgroup of G, then $N_G(S)$ is G-completely reducible and if further S^0 is central in S, then $(G, N_G(S))$ is a reductive pair.*

Proof. (a). Let H be the union of the components of G that meet $C_G(S)$. Then H is a finite-index subgroup of G, so $C_G(S)$ is G-cr if and only if it is H-cr, by [4, Prop. 2.12]. Hence we can assume that $G = H$. Replacing S by $S/C_S(G)$, we can also assume that S acts faithfully on G. Since $C_G(S)$ meets every component of G, it follows that S acts faithfully on G^0. The completely reducible representation $S \to \mathrm{GL}(\mathfrak{g})$ has finite kernel by Lemma 3.6, so S is reductive.

If $\mathrm{Char}\, k = 0$, then, as S is reductive, it is linearly reductive. So in this case all we have to show is that $C_G(S)$ is reductive. Since $C_G(S)^0 = C_{G^0}(S)^0$, this follows immediately from [26, Prop. 10.1.5].

Now assume that $\mathrm{Char}\, k = p > 0$. By Lemma 2.3, S admits an ascending chain of finite subgroups $S_1 \subseteq S_2 \subseteq \cdots$ such that their union is Zariski dense in S. From this it follows that for some $i \geq 1$, $\mathfrak{g}$ is a semisimple S_i-module (that is, the image of S_i in $\mathrm{GL}(\mathfrak{g})$ is $\mathrm{GL}(\mathfrak{g})$-cr — cf. [3, Lem. 2.10]) and we have $C_G(S_i) = C_G(S)$ and $\mathfrak{c}_{\mathfrak{g}}(S_i) = \mathfrak{c}_{\mathfrak{g}}(S)$. So, after replacing S by S_i, we may assume that S is finite. Put $G' = G \rtimes S$. Clearly, G' is reductive. Furthermore, $\mathfrak{g}' = \mathrm{Lie}\, G' = \mathfrak{g}$ is a semisimple S-module, and $\mathrm{Lie}\, C_{G'}(S) = \mathfrak{c}_{\mathfrak{g}'}(S)$ by construction. Therefore, S is G'-cr, by Theorem 1.6. Since S is G'-cr, $C_{G'}(S)$ is G'-cr, by [3, Cor. 3.17]. So the normal subgroup $C_G(S)$ of $C_{G'}(S)$ is also G'-cr, [3, Thm. 3.10]. Now $C_G(S)$ is G-cr, by [4, Prop. 2.12].

Since $\mathfrak{g}$ is a semisimple S-module, it has a direct sum decomposition into S-isotypic summands. By hypothesis, $\mathrm{Lie}\, C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$, so $\mathrm{Lie}\, C_G(S)$ is the trivial S-isotypic summand. Hence there is a unique S-stable complement to $\mathrm{Lie}\, C_G(S)$ in $\mathfrak{g}$, namely, the sum $\mathfrak{m}$ of the non-trivial S-isotypic summands. The uniqueness of $\mathfrak{m}$ implies that it is also $C_G(S)$-stable, so it follows that $(G, C_G(S))$ is a reductive pair.

(b). Now assume that S is a subgroup of G. Then S is G-cr by Theorem 1.6, so $N_G(S)$ is G-cr [3, Thm. 3.14]. Moreover, $N_G(S)/SC_G(S)$ is finite by [22, Lem. 6.8]. Since $S^0 \subseteq C_G(S)$ by assumption, $N_G(S)/C_G(S)$ is finite. It follows that $\mathrm{Lie}\, N_G(S) = \mathrm{Lie}\, C_G(S)$. By the uniqueness of the subspace $\mathfrak{m}$ of $\mathfrak{g}$ from the proof of part (a) above, $\mathfrak{m}$ is also $N_G(S)$-stable. Hence $(G, N_G(S))$ is a reductive pair. $\square$

The following is immediate by Theorem 1.4, Corollary 2.12 and Proposition 3.7(a). Note that it applies in particular when S is a torus, so that $C_G(S)$ is an R-Levi subgroup of G [3, Cor. 6.10].

Corollary 3.8. *Let G and S be as in the statement of Proposition 3.7. Then every subgroup of $C_G(S)$ which is separable in G is separable in $C_G(S)$ and every subalgebra of $\mathfrak{c}_{\mathfrak{g}}(S)$ which is separable in $\mathfrak{g}$ is separable in $\mathfrak{c}_{\mathfrak{g}}(S)$.*

For a regular reductive subgroup H of G the next lemma gives a useful criterion for (G, H) to be a reductive pair (cf. [3, Ex. 3.33]).

Lemma 3.9. *Let T be a maximal torus of G and let H be a reductive subgroup of G containing T. Assume that $\Psi(H) = \Psi(H,T)$ is a closed subsystem of $\Psi = \Psi(G,T)$. Then (G, H) is a reductive pair.*

Proof. Let $\mathfrak{m}$ be the sum of the root spaces $\mathfrak{u}_\alpha$ with $\alpha \notin \Psi(H)$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. By the conjugacy of the maximal tori in H^0, we have $H = H^0 N_H(T)$. Since $N_H(T)$ permutes the root spaces in $\mathfrak{h}$ and also those outside $\mathfrak{h}$, $N_H(T)$ stabilizes $\mathfrak{m}$. So all we have to show is that $\mathfrak{m}$ is stable under the U_β, for $\beta \in \Psi(H)$.

Let $\alpha \in \Psi \setminus \Psi(H)$ and $\beta \in \Psi(H)$. If $\gamma = \alpha + i\beta$ is a root for some integer $i \geq 0$, then we must have $\gamma \notin \Psi(H)$, as otherwise $\alpha = \gamma - i\beta \in \Psi(H)$, because $\Psi(H)$ is closed and symmetric; see [9, Ch. 6, Prop. 23(iii)]. Now let $u \in U_\beta$. Then, by [6, Lem. 5.2], $\operatorname{Ad} u(\mathfrak{u}_\alpha) \subseteq \bigoplus \mathfrak{u}_{\alpha+i\beta}$, where the sum is over all integers $i \geq 0$ such that $\alpha + i\beta$ is a root. By the above, this sum is contained in $\mathfrak{m}$. □

Remarks 3.10. (i). Note that if S is a linearly reductive group acting on G by automorphisms, then the conditions of Proposition 3.7(a) are satisfied; see [26, Lem. 4.1].

(ii). We can apply Proposition 3.7(b) and Theorem 1.4 to prove that if S is a linearly reductive subgroup of G with S^0 central in S and H is a G-cr subgroup of $N_G(S)$ such that H is separable in G, then H is $N_G(S)$-cr. Here the separability condition cannot be removed; see Example 7.21.

(iii). Note that the hypothesis on S^0 in Proposition 3.7(b) cannot be removed. For example, suppose $p = 3$ and let $G = (k^* \times k^* \times k^*) \rtimes (C_3 \times C_2)$, where the cyclic group C_3 acts on $k^* \times k^* \times k^*$ by a cyclic permutation of the factors and the cyclic group $C_2 = \{1, a\}$ acts on $k^* \times k^* \times k^*$ by $a \cdot (x, y, z) = (x^{-1}, y^{-1}, z^{-1})$. Let S be the linearly reductive subgroup ΔC_2, where Δ is the diagonal inside $k^* \times k^* \times k^*$. Then $N_G(S)^0 = \Delta$ and $C_3 \subseteq N_G(S)$, so $(G, N_G(S))$ is not a reductive pair because $\operatorname{Lie} \Delta$ does not admit a C_3-stable complement in $\operatorname{Lie}(k^* \times k^* \times k^*)$.

(iv). We observe that Slodowy's example [34, I.3(7)] is a special case of Proposition 3.7(a), namely when $G = \mathrm{GL}(V)$ and $C_G(S)$ is a Levi subgroup of G.

(v). In the special case of Proposition 3.7 when S is linearly reductive and G is connected, Richardson showed in [26, Prop. 10.1.5] that $C_G(S)$ is reductive and, if S is a subgroup of G, $N_G(S)$ is reductive.

(vi). The converse of Corollary 3.8 is false; see Proposition 7.11.

We give another application of Proposition 3.7.

Example 3.11. Suppose that $p = 2$. Let G be simple of type D_4 and let S be the group of order 3 generated by the triality graph automorphism of G. Then $K = C_G(S)^0$ is of type G_2. Since S is linearly reductive, Proposition 3.7(a) implies that (G, K) is a reductive pair. In Section 7, we construct a subgroup H of K isomorphic to S_3 which is not separable in K; see Proposition 7.11. It follows from Theorem 1.4 that H is also non-separable in G. In addition, by Lemma 7.10(a), H is K-cr and thus, thanks to [3, Cor. 3.21], H is also G-cr.

This example also gives rise to a non-separable subgroup in the exceptional group of type F_4 as follows. Let G' denote this group; then, since D_4 is a closed subsystem of the root system of type F_4 (the D_4 subsystem consists of the long roots in the F_4 system), Lemma 3.9 implies that (G', G) is a reductive pair. It follows from Theorem 1.4 that H is also non-separable in the group G'.

4. The adjoint module and complete reducibility

In the proof of Theorem 1.6, the hypothesis of separability is used only for a rather coarse dimension-counting argument, so it is natural to ask whether it can be removed. This is a more subtle problem than it at first appears.

Our first result shows that we can remove the separability assumption from H in Theorem 1.6 under extra hypotheses on H.

Theorem 4.1. *Suppose that H is a subgroup of G such that H acts semisimply on $\mathfrak{m}_{\mathrm{ad}}$ for every reductive subgroup M of G that contains H and a maximal torus of G (this includes the case $M = G$). Then H is G-completely reducible.*

Proof. Suppose P is an R-parabolic subgroup of G containing H, let T be a maximal torus of P and let $\lambda \in Y(T)$ such that $P = P_\lambda$. By Remark 2.5, we can assume that P^0 is proper in G^0. This implies that λ is non-central in G^0 [3, Lem. 2.4].

First assume that H acts semisimply on $\mathfrak{g}$ and that $\mathfrak{z}(\mathfrak{g}) = \{0\}$. We then show that there exists a reductive subgroup M of G such that $\dim M < \dim G$ and M contains H, $\lambda(k^*)$ and a maximal torus of G. After that we prove the statement of the theorem using induction on the dimension of G.

Put $\mathfrak{n} = \operatorname{Lie} R_u(P)$, $\mathfrak{l}_\lambda = \operatorname{Lie} L_\lambda$ and $\mathfrak{s} = \operatorname{Lie} \lambda(k^*)$. Since L_λ centralizes $\lambda(k^*)$ and normalizes $R_u(P)$, $\lambda(k^*)R_u(P)$ is normalized by L_λ. Therefore, $\lambda(k^*)R_u(P)$ is a normal subgroup of P. So $\operatorname{Lie}(\lambda(k^*)R_u(P)) = \mathfrak{n} \oplus \mathfrak{s}$ is an H-submodule of $\mathfrak{g}$, since H is contained in P. Now $\mathfrak{n}$ is an H-submodule of $\mathfrak{n} \oplus \mathfrak{s}$, so, since $\mathfrak{g}$ is a semisimple H-module, there exists a 1-dimensional H-submodule $\mathfrak{s}_1$ of $\mathfrak{n} \oplus \mathfrak{s}$ which is a direct H-complement to $\mathfrak{n}$.

As L_λ acts trivially on $\mathfrak{z}(\mathfrak{l}_\lambda)$, $(\mathfrak{n} \oplus \mathfrak{z}(\mathfrak{l}_\lambda))/\mathfrak{n}$ is a trivial L_λ-module. By [7, Prop. I.3.17], we have $\operatorname{Ad}(u)(x) - x \in [\mathfrak{l}, \mathfrak{n}] \subseteq \mathfrak{n}$ for all $u \in R_u(P)$ and all $x \in \mathfrak{p}$, so $(\mathfrak{n} \oplus \mathfrak{z}(\mathfrak{l}_\lambda))/\mathfrak{n}$ is also a trivial $R_u(P)$-module and therefore a trivial P-module. So $\mathfrak{s}_1$ must be a trivial H-module. Let $x \in \mathfrak{s}_1$ be non-zero. Clearly, H fixes the nilpotent and semisimple parts of x. By [7, Thm. III.10.6], $\mathfrak{n}$ is the set of nilpotent elements of $\mathfrak{n} \oplus \mathfrak{s}$. So x has non-zero semisimple part and we may assume that x is semisimple. Thus we have found a non-zero semisimple element of $\operatorname{Lie} P$ that is fixed by H and $\lambda(k^*)$. After conjugating λ and T by the same element of P, we may assume that $x \in \operatorname{Lie} T$.

Put $M := C_G(x)$. This is a reductive subgroup of G which contains T, H and $\lambda(k^*)$. Furthermore, $\dim M < \dim G$, since otherwise $G^0 \subseteq M$, which is impossible, because $\mathfrak{c}_\mathfrak{g}(G^0) = \mathfrak{z}(\mathfrak{g}) = \{0\}$. So M has the required properties.

To prove the assertion of the theorem, we pass to the adjoint group. Let $\pi_G : G \to G_{\mathrm{ad}}$ be the natural morphism and let $\tilde{\lambda} = \pi_G \circ \lambda \in Y(G_{\mathrm{ad}})$. By our hypothesis, $\pi_G(H)$ acts semisimply on $\mathfrak{g}_{\mathrm{ad}}$. Note that $\tilde{\lambda}$ is non-trivial. If we apply the above argument to G_{ad}, $\pi_G(H)$ and $\tilde{\lambda}$, we get a reductive subgroup M' of G_{ad} such that $\dim M' < \dim G_{\mathrm{ad}}$ and M' contains $\pi_G(T)$, $\pi_G(H)$ and $\tilde{\lambda}(k^*)$. But then $M := \pi_G^{-1}(M')$ is a reductive subgroup of G such that $\dim M < \dim G$ and M contains T, H and $\lambda(k^*)$. Clearly, M and H satisfy the same assumptions as G and H. By the induction hypothesis, H is M-cr, so there exists $u \in P_\lambda(M)$ such that $H \subseteq uL_\lambda(M)u^{-1}$. But $R_u(P_\lambda(M)) = R_u(P_\lambda) \cap M$ and similarly for $L_\lambda(M)$. So H is contained in the Levi subgroup $uL_\lambda u^{-1}$ of P. $\square$

The following example indicates the sort of problem that can arise without the assumptions on H made in Theorem 4.1. First we need some terminology. If G is connected and simple of type A_{n-1}, then either $\operatorname{Lie} G \cong \mathfrak{sl}_n$, $\operatorname{Lie} G \cong \mathfrak{pgl}_n$, or $p^2 | n$ and $\operatorname{Lie} G$ is of *intermediate type*; see [12, Table 1]. In the latter case, $\mathfrak{g}$ is the direct sum of its centre, which is 1-dimensional, and its derived algebra, which is isomorphic to $\mathfrak{psl}_n := \mathfrak{sl}_n/(k \cdot \mathrm{id})$.

Example 4.2. Let p be a prime, put $n = p^2$ and let G be the simple algebraic group of type A_{n-1} whose character group is the lattice that is strictly between the root lattice and the weight lattice (the quotient group of the latter two lattices is a cyclic group of order p^2). Then $\mathfrak{g}$ is of intermediate type. Thus $\mathfrak{g}$ is the direct sum of two simple G-modules and is therefore a semisimple G-module. But the only proper non-zero ideal of $\mathfrak{g}_{\mathrm{ad}} \cong \mathfrak{pgl}_n$ is its derived subalgebra, which is of dimension

$n^2 - 2$ ([12, Table 1]). Since every G-submodule of $\mathfrak{g}_{\rm ad}$ is an ideal, it follows that G does not act semisimply on $\mathfrak{g}_{\rm ad}$.

Remarks 4.3. (i). Example 4.2 shows that it is possible for a subgroup H to act semisimply on $\mathfrak{g}$, but not on $\mathfrak{g}_{\rm ad}$. Thus the argument in the proof of Theorem 4.1 cannot be used to extend Theorem 1.6 to the non-separable case.

(ii). Suppose G is semisimple (so $Z(G)$ is finite), and H is a subgroup of G which acts semisimply on $\mathfrak{g}$, but H is not separable in G. Observe that H might be "trivially" non-separable in G, in the sense that $\mathfrak{g}$ has non-zero centre so that G is not even separable in itself. One might hope to deal with such possibilities by passing to the adjoint form $G_{\rm ad}$ of G, but Example 4.2 shows that if we do this, the image of H in $G_{\rm ad}$ may fail to act semisimply on $\mathfrak{g}_{\rm ad}$.

Our next result shows that we can also remove the separability assumption in Theorem 1.6 by strengthening the conditions on G, rather than on H (as in Theorem 4.1). In contrast to Theorem 1.7, this next result does not impose any characteristic restrictions stemming from simple factors of G of type A_n.

Theorem 4.4. *Assume that G is connected, p is good for G, and no simple factor of type A_n of the derived group DG of G has Lie algebra of intermediate type. Let H be a subgroup of G which acts semisimply on* $\operatorname{Lie} DG$. *Then H is G-completely reducible.*

Proof. Since $Z(G)^0$ acts trivially on $\mathfrak{g}$ and since it is contained in any Levi subgroup of any parabolic subgroup of G, we may assume that $Z(G)^0 \subseteq H$. Then $H = Z(G)^0(H \cap DG)$. Applying [3, Lem. 2.12] to the isogeny $Z(G)^0 \times DG \to G$ and the projection $Z(G)^0 \times DG \to DG$, we see that we may replace G by DG and H by $H \cap DG$.

Let $G_1, G_2, \ldots, G_r$ be the simple factors of G and let $\mu : \Pi_i G_i \to G$ be the isogeny given by multiplication. Denote the projection $G \to G_i$ by π_i and put $\mathfrak{g}_i = \operatorname{Lie} G_i$. Note that $\mathfrak{g}_i$ is H-semisimple, since it is a G-submodule of $\mathfrak{g}$. It is easily checked that for each i the set of automorphisms of $\mathfrak{g}_i$ given by H is the same as that given by $\pi_i(\mu^{-1}(H))$. So $\pi_i(\mu^{-1}(H))$ acts semisimply on $\mathfrak{g}_i$ for each i. Thanks to [3, Lem. 2.12], we may now assume that G is simple.

First assume that G is not of type A. Then H is G-cr by Theorem 1.7.

Now assume that G is of type A_{n-1}. Since $\mathfrak{g}$ is not of intermediate type, the isogenies $\varphi : \mathrm{SL}_n \to G$ and $\psi : G \to \mathrm{PGL}_n$ cannot both be inseparable. So $\varphi^{-1}(H)$ acts semisimply on $\mathfrak{sl}_n$ or $\psi(H)$ acts semisimply on $\mathfrak{pgl}_n$. By [3, Lem. 2.12(ii)(b)], we may assume that $G \cong \mathrm{SL}_n$ or that $G \cong \mathrm{PGL}_n$.

First assume that $G = \mathrm{SL}_n$. We have $G_{\rm ad} = \mathrm{PGL}_n$ and $\mathfrak{g}_{\rm ad} = \mathfrak{pgl}_n$. The trace form on $\mathfrak{gl}_n$ is non-degenerate and induces a non-degenerate GL_n-invariant pairing between $\mathfrak{sl}_n$ and $\mathfrak{pgl}_n$. This shows that $\mathfrak{pgl}_n \cong \mathfrak{sl}_n^*$ as GL_n-modules and therefore that H acts semisimply on $\mathfrak{g}_{\rm ad}$. More generally, thanks to [8], if M is any reductive subgroup of G containing a maximal torus of G, then M^0 is a Levi subgroup of G (see also [9, Ex., Ch. VI, §4.4]). Regarding G as a subgroup of GL_n, we can write $M^0 = (\prod_{i=1}^r \mathrm{GL}_{n_i}) \cap \mathrm{SL}_n$ and $\mathfrak{m} = (\bigoplus_{i=1}^r \mathfrak{gl}_{n_i}) \cap \mathfrak{sl}_n$ for some n_i. Since the restriction of the trace form of $\mathfrak{gl}_n$ to $\bigoplus_{i=1}^r \mathfrak{gl}_{n_i}$ is non-degenerate — it is the direct sum of the trace forms of the $\mathfrak{gl}_{n_i}$ — the orthogonal complement of $\mathfrak{m}$ in $\bigoplus_{i=1}^r \mathfrak{gl}_{n_i}$ is 1-dimensional and is therefore equal to $k \cdot \mathrm{id}_n$. So $\mathfrak{m}^* \cong (\bigoplus_{i=1}^r \mathfrak{gl}_{n_i})/(k \cdot \mathrm{id}_n)$ as M-modules. Now $\mathfrak{m}_{\rm ad} = \bigoplus_{i=1}^r \mathfrak{pgl}_{n_i}$, which is isomorphic to a quotient of $\mathfrak{m}^* = (\bigoplus_{i=1}^r \mathfrak{gl}_{n_i})/(k \cdot \mathrm{id}_n)$. Thus if H acts semisimply on $\mathfrak{g}$, then H acts semisimply on

$\mathfrak{m}_{\mathrm{ad}}$ for any reductive subgroup M which contains H and a maximal torus of G. The result now follows from Theorem 4.1.

Finally, assume that G is isomorphic to PGL_n. Let $\varphi : \mathrm{SL}_n \to \mathrm{PGL}_n$ be the canonical projection. Since $\mathfrak{pgl}_n \cong \mathfrak{sl}_n^*$ as GL_n-modules, $\varphi^{-1}(H)$ acts semisimply on $\mathfrak{sl}_n$. The desired result follows from the previous case and [3, Lem. 2.12(ii)(b)]. □

We record two important special cases of Theorem 4.4.

Corollary 4.5. *Assume that G is connected, p is good for G, and DG is either adjoint or simply connected. Let H be a subgroup of G which acts semisimply on* $\mathrm{Lie}\, DG$. *Then H is G-completely reducible.*

Remarks 4.6. (i). Note that Theorem 4.4 applies in cases when Theorem 1.6 does not. For example, let $G = \mathrm{SL}_2(k)$, where k has characteristic 2, let T be a maximal torus of G, and set $H = N_G(T)$. Then $\mathfrak{g}$ is semisimple as an H-module, so Theorem 4.4 applies, but, as H is not separable in G (cf. [21, Ex. 3.4(b)]), Theorem 1.6 does not apply. We can also take $G = \mathrm{GL}_2(k)$, k of characteristic 2, and $H = N_G(T)$, where T is a maximal torus of G. Then $\mathrm{Lie}\, DG = \mathfrak{sl}_2$ is a semisimple H-module, but $\mathfrak{g} = \mathfrak{gl}_2$ is not semisimple as an H-module.

(ii). Let H be a group and let $\rho : H \to \mathrm{GL}(V)$ be a finite-dimensional representation of H. We have $V \otimes V^* \cong \mathfrak{gl}(V)$ as $\mathrm{GL}(V)$-modules and therefore also as H-modules. Furthermore, V is H-semisimple if and only if $\rho(H)$ is $\mathrm{GL}(V)$-cr. So, by Theorem 4.4 (or Theorem 1.6), V is H-semisimple if $V \otimes V^*$ is H-semisimple. This result is a special case of a theorem of Serre; see [29, Thm. 3.3].

(iii). We do not know whether the assumptions other than $\mathfrak{g}$ being H-semisimple are necessary in Theorem 4.4. To show that the assumption on the simple type A factors can be removed, it would suffice to show that, under the assumption that $p^2|n$, a subgroup H of GL_n which acts semisimply on $\mathfrak{psl}_n$ is GL_n-cr — that is, acts semisimply on the natural module k^n.

(iv). It is easy to prove that k^n is H-semisimple if $\mathfrak{pgl}_n$ (equivalently, $\mathfrak{sl}_n$) is H-semisimple. The arguments are straightforward modifications of the arguments at the end of the proof of Weyl's theorem in [13, Thm. 6.3] and work for an arbitrary field k. So one can prove Theorem 4.4 without using Theorem 4.1.

We finish this section with an example which illustrates that the converse of Theorem 1.6 is false.

Example 4.7. In [32, Cor. 5.5], Serre shows that $\mathfrak{g}$ is a semisimple H-module if and only if H is G-completely reducible, under the assumption that $p > 2h - 2$, where h is the Coxeter number of G. Theorem 1.7 improves the bound for the forward implication of this equivalence. However, Serre's bound is sharp for the reverse implication, as the following example (due to some unpublished calculations of Serre) shows; see also [3, Rem. 3.43(iii)].

Let H be an exceptional simple group of adjoint type and suppose $p \leq 2h - 2$, where h denotes the Coxeter number of H. Then, with the possible exception of type G_2 in characteristic 5, Serre's construction yields a simple subgroup K of H of type A_1 such that K is H-irreducible, but K does not act semisimply on $\mathfrak{h}$ — that is, K is not G-completely reducible, where $G := \mathrm{GL}(\mathfrak{h})$.

This example also illustrates that the converse of the second assertion of Theorem 1.4 is not true in general, even when K is separable in G and (G, H) is a reductive pair. For we can choose p and H above such that p is good for H; then $(\mathrm{GL}(\mathfrak{h}), H)$

is a reductive pair (cf. [3, Ex. 3.37]), since the Killing form on $\mathfrak{h}$ is non-degenerate [25, §5]. It then follows from Theorem 1.4 that K is separable in H, since K is separable in $\mathrm{GL}(\mathfrak{h})$.

5. Closed orbits and separability

In this section we generalize two results of R.W. Richardson [26, Thm. A, Thm. C], which we are then able to apply to G-complete reducibility. We require two preliminary results, which allow us to relax the hypotheses in the original theorems. Armed with these lemmas, it is quite straightforward to adapt Richardson's proofs to our setting.

Our first result is an analogue of [26, Prop. 6.1], which Richardson proves for S linearly reductive and G connected, using non-commutative cohomology of algebraic groups. It should be noted that, although our result is more general than Richardson's, in order to apply it to linearly reductive groups one needs to know that they are always completely reducible in any ambient reductive group ([3, Lem. 2.6]), and the proof of this depends on the argument that Richardson uses.

Lemma 5.1. *Let G be a normal subgroup of the reductive group G' and suppose that S is a G'-completely reducible subgroup of G'. Then for every R-parabolic subgroup P of G normalized by S, there exists an R-Levi subgroup of P normalized by S.*

Proof. Note that since G is normal in G', G is reductive, and G^0 is a connected reductive normal subgroup of G'. We first show that it is enough to prove the result when G is connected. If P is an R-parabolic subgroup of G normalized by S, then P^0 is a parabolic subgroup of G^0 normalized by S. If L is a Levi subgroup of P^0 normalized by S, then $L = M^0$ for some R-Levi subgroup M of P [3, Cor. 6.8]. Let T be a maximal torus of M; then $T \subseteq L$. Since S normalizes P, it follows that sMs^{-1} is an R-Levi subgroup of P for any $s \in S$. Moreover, since S normalizes L, we see that $T \subseteq L = sLs^{-1} \subseteq sMs^{-1}$. This shows that $M = sMs^{-1}$, for every $s \in S$, by [3, Cor. 6.5], and therefore S normalizes M. Thus, if the result is true for G^0, then it is true for G.

Now suppose G is connected, and let $H = C_{G'}(G)^0$. Since G is normal in G', we have $G'^0 = GH$, by [22, Lem. 6.8]. Let P be a parabolic subgroup of G normalized by S. Then $P \times H$ is a parabolic subgroup of $G \times H$, and it follows from [7, Prop. IV.11.14(1)] applied to the multiplication map $f\colon G \times H \to G'^0$ that PH is a parabolic subgroup of G'^0. As S normalizes P, S normalizes PH, so $S \subseteq N_{G'}(PH)$, which is an R-parabolic subgroup of G' by [22, Prop. 5.4(a)]. Now S is G'-cr by hypothesis, so S is contained in some R-Levi subgroup M of $N_{G'}(PH)$, so S normalizes M^0, which is a Levi subgroup of $N_{G'}(PH)^0 = PH$. By [3, Lem. 6.15(ii)], $f^{-1}(PH)$ is a parabolic subgroup of $G \times H$ with Levi subgroup $f^{-1}(M^0)$.

If $(g,h) \in f^{-1}(PH)$, then $gh = g'h'$ for some $g' \in P$ and some $h' \in H$, so $(g')^{-1}g = h'h^{-1} \in G \cap H \subseteq Z(G) \subseteq P$, so $g \in P$. It follows that $f^{-1}(PH) = P \times H$. Hence $f^{-1}(M^0) = L \times H$ for some Levi subgroup L of P and hence $M^0 = LH$. A similar calculation shows that $L = M^0 \cap G$, so S normalizes L and we are done. $\square$

Lemma 5.2. *Let S be a group and let $f : V \to W$ be a surjective homomorphism of finite-dimensional S-modules over some field. If V/V^S has no trivial composition factor, then $f(V^S) = W^S$.*

Proof. Since f is surjective, it induces a surjective homomorphism $V/V^S \to W/f(V^S)$ of S-modules. Since V/V^S has no trivial composition factor, neither does $W/f(V^S)$. In particular, $W/f(V^S)$ does not have any non-zero S-fixed points. So W^S must be equal to $f(V^S)$. □

Remark 5.3. If V is S-semisimple — in particular, if S is a linearly reductive algebraic group — then V/V^S has no trivial composition factor, so Lemma 5.2 applies.

We can now state our generalization of Richardson's two results [26, Thm. A, Thm. C]. Observe that if S is a subgroup of G, then the condition on S in Theorem 5.4(b)(ii) is just that of separability of S in G.

Theorem 5.4. *Let G be a normal subgroup of the reductive group G'. Let S be a subgroup of G' such that $G' = GS$. Let G' act on a variety X and let $x \in X^S$. Let $\mathcal{O}$ denote the unique closed G-orbit in the closure of $G \cdot x$.*

- (a) *Suppose that X is affine and S is G'-completely reducible. Then $C_G(S)$ is G-completely reducible and there exists $\lambda \in Y(C_G(S))$ such that $\lim_{a\to 0} \lambda(a)\cdot x$ exists and is a point of $\mathcal{O}$. In particular, $G\cdot x$ is closed if $C_G(S)\cdot x$ is closed.*
- (b) *Suppose that*
  - (i) $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$ *does not have any trivial S-composition factors;*
  - (ii) $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$.

  Then each irreducible component of $G\cdot x \cap X^S$ is a single $C_G(S)^0$-orbit. In particular, $C_G(S)\cdot x$ is closed if $G\cdot x$ is closed.

Proof. (a). Suppose that S is G'-cr. Then $C_G(S)$ is G-cr, by the same arguments as in the proof of Proposition 3.7(a). If $G \cdot x$ is closed, then we can take λ to be the zero cocharacter, so assume that $G \cdot x$ is not closed. Then $G^0 \cdot x$ is not closed. Moreover, the unique closed G-orbit $\mathcal{O}$ in the closure of $G\cdot x$ contains the unique closed G^0-orbit in the closure of $G^0 \cdot x$. Thus, since $Y(G) = Y(G^0)$ and $Y(C_G(S)) = Y(C_{G^0}(S))$, we may replace G by G^0 and assume that G is connected.

We can now follow the first part of Richardson's original proof in [26, Sec. 8] word for word to give the result, simply replacing references to [26, Prop. 6.1] with our Lemma 5.1.

(b). Now assume that $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$ does not have any trivial S-composition factors and that $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$. We can write $G\cdot x$ as a finite union $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$ of G^0-orbits. Each $\mathcal{O}_i$ is a G-translate of $G^0\cdot x$, from which we deduce that the $\mathcal{O}_i$ are the irreducible components of $G\cdot x$. We may assume that $\mathcal{O}_1, \ldots, \mathcal{O}_r$ ($r \le n$), are the G^0-orbits in $G\cdot x$ that meet X^S. Then $G\cdot x\cap X^S = (\mathcal{O}_1\cap X^S)\cup\cdots\cup(\mathcal{O}_r\cap X^S)$, a disjoint union of closed subsets of $G\cdot x$. This shows that, after replacing G' by G^0S, we may assume that G is connected.

Note that $G\cdot x$ is S-stable. Put $Y = (G\cdot x)^S = G\cdot x\cap X^S$. By the argument at the end of the proof of [26, Thm. A], we may assume that the orbit map $G \to G\cdot x$ of x is separable. Let $y \in Y$. Then the orbit map of y is separable; that is, its differential $\psi : \mathfrak{g} \to T_y(G\cdot y)$ is surjective. Since the orbit map of y is S-equivariant, ψ is S-equivariant. By Lemma 5.2, we have $\psi(\mathfrak{c}_{\mathfrak{g}}(S)) = (T_y(G\cdot y))^S$. Clearly, $T_y(Y) \subseteq (T_y(G\cdot x))^S$, so $T_y(Y) \subseteq \psi(\mathfrak{c}_{\mathfrak{g}}(S))$. Since $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$, the Tangent Space Lemma [26, Lem. 3.1] gives the first assertion of (b).

It now follows that $C_G(S)^0 \cdot x$, and therefore $C_G(S) \cdot x$, is closed in $G \cdot x$. So if $G \cdot x$ is closed, then $C_G(S) \cdot x$ is closed. □

Remarks 5.5. (i). Assume that S is a separable subgroup of G which acts semisimply on $\mathfrak{g}$. Then S is G-cr by Theorem 1.6. So parts (a) and (b) of Theorem 5.4 both apply. Thus in this case $C_G(S) \cdot x$ is closed if and only if $G \cdot x$ is closed.

(ii). Suppose that S is linearly reductive. Then conditions (i) and (ii) of Theorem 5.4(b) are satisfied (see [26, Lem. 4.1]), and S is G'-cr [3, Lem. 2.6]; the special case of Theorem 5.4 for S linearly reductive is precisely Richardson's original theorem [26, Thm. C] (clearly, the hypothesis in Theorem 5.4 that $G' = GS$ is harmless).

(iii). It follows from the proof of Theorem 5.4(b) that [26, Thm. A] holds with the hypothesis that S acts semisimply on $\mathfrak{g}$ replaced by the weaker hypothesis of Theorem 5.4(b)(ii).

(iv). The result in Theorem 5.4(a) that the G-orbit is closed if the $C_G(S)$-orbit is closed is also a slight generalization of part of [2, Thm. 4.4].

(v). Theorem 5.4(b) does not hold for an arbitrary G'-completely reducible subgroup S. In fact it does not even hold when S is a G-completely reducible subgroup of G: in [4, Prop. 3.9] it is shown that if S is a G-completely reducible subgroup of G and K is a subgroup of $C_G(S)$, then K is $C_G(S)$-completely reducible if and only if KS is G-completely reducible. Now [4, Examples 5.1, 5.3, 5.5] give instances of commuting G-completely reducible subgroups K and S such that KS is not G-completely reducible, whence K is not $C_G(S)$-completely reducible. If the field k is large enough (cf. [3, Rem. 2.9]), we can pick an n-tuple $(k_1, \dots, k_n) \in G^n$ topologically generating K for some n. Then by Theorem 2.9, $G \cdot (k_1, \dots, k_n)$ is closed in G^n, but $C_G(S) \cdot (k_1, \dots, k_n)$ is not.

Corollary 5.6. *Let G, G', S, X and $x \in X^S$ be as in Theorem* 5.4*, and suppose X is affine. Suppose that*

(i) *the adjoint representation of S on $\mathfrak{g}$ is semisimple;*
(ii) $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$.

Then S is reductive, $C_G(S)$ is G-completely reducible and $G \cdot x$ is closed if and only if $C_G(S) \cdot x$ is closed.

Proof. Since $GS = G'$, the quotient $S/(S \cap G^0)$ is isomorphic to a finite-index subgroup of G'/G^0, and hence is reductive. Now $S \cap G^0$ is normal in S, so $S \cap G^0$ acts semisimply on $\mathfrak{g}$ by Clifford's Theorem. The kernel N of the $(S \cap G^0)$-action on $\mathfrak{g}$ is a subgroup of the diagonalizable group $Z(G^0)$ and is therefore reductive. Since N and $(S \cap G^0)/N$ are reductive, $S \cap G^0$ is reductive, and it follows that S is reductive.

Suppose $\operatorname{Char} k = 0$. Then S is G'-cr, since S is reductive. Therefore, the hypotheses of Theorem 5.4(a) and (b) are satisfied.

Now suppose $\operatorname{Char} k = p > 0$. By the same arguments as in the proof of Proposition 3.7(a), we may assume that S is finite and deduce that then S is G'-cr. Thus, the hypotheses of Theorem 5.4(a) and (b) are again satisfied. □

Now we translate our results in terms of G-complete reducibility.

Proposition 5.7. *Let G be a normal subgroup of the reductive group G'. Let S be a subgroup of G' such that $G' = GS$. Let $K \subseteq H$ be subgroups of G such that $H^0 = C_G(S)^0$ and S normalizes K.*

(a) *Suppose that S is G'-completely reducible. Then H is reductive and K is G-completely reducible if it is H-completely reducible.*

(b) *Suppose that*

(i) $\mathfrak{g}/\mathfrak{c}_{\mathfrak{g}}(S)$ *does not have any trivial S-composition factors;*

(ii) $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$*;*

(iii) $C_G(S)$ *is reductive.*

Then H is reductive and K is H-completely reducible if it is G-completely reducible.

Proof. In case (a), $C_G(S)$ is reductive by Theorem 5.4(a) and in case (b), this is true by assumption. Since $H^0 = C_G(S)^0$, it follows that H is reductive.

Clearly, we can now assume that $\operatorname{Char} k = p > 0$. By the argument in the proof of Proposition 3.7(a), we may also assume that S is finite. As in [3, Lem. 2.10], we can replace K with a subgroup K' that is topologically generated by some $k_1, \dots, k_n$ with the property that for any $\lambda \in Y(G)$, we have $K \subseteq P_\lambda$ if and only if $K' \subseteq P_\lambda$, and $K \subseteq L_\lambda$ if and only if $K' \subseteq L_\lambda$. Thus K is G-cr (respectively H-cr) if and only if K' is G-cr (respectively H-cr).

Since S is finite, we may assume, by replacing $(k_1, \dots, k_n)$ with a larger tuple if necessary, that S permutes the k_i and therefore that S also normalizes K'. Since $H^0 = C_G(S)^0$, we have that $H\cdot(k_1, \dots, k_n)$ is closed if and only if $C_G(S)\cdot(k_1, \dots, k_n)$ is closed.

Let G' act on G^n by simultaneous conjugation. The symmetric group S_n acts naturally on G^n, and the G'-action commutes with this action. Set $X = G^n/S_n$ and let $\pi : G^n \to X$ be the natural map; the fibres of π are precisely the S_n-orbits (see [1, Sec. 2], for example). For any subgroup M of G' and any $(g_1, \dots, g_n) \in G^n$, we have that $M \cdot (g_1, \dots, g_n)$ is closed in G^n if and only if $M \cdot \pi((g_1, \dots, g_n))$ is closed in X. Put $x = \pi((k_1, \dots, k_n))$. Then $x \in X^S$. Assertions (a) and (b) now follow from Theorem 2.9 and from Theorem 5.4(a) and (b), respectively. □

The next corollary is a generalization of [3, Cor. 3.21]. Note that the hypotheses on S in Corollary 5.8 are satisfied if S is linearly reductive.

Corollary 5.8. *Let S be an algebraic group acting on G by automorphisms. Suppose that S acts semisimply on $\mathfrak{g}$ and $\mathfrak{c}_{\mathfrak{g}}(S) = \operatorname{Lie} C_G(S)$. Let $K \subseteq H$ be subgroups of G such that $H^0 = C_G(S)^0$ and S normalizes K. Then H is reductive and K is G-completely reducible if and only if K is H-completely reducible.*

Proof. By Proposition 3.7(a), $C_G(S)$ is reductive. So H is also reductive. We can now assume that $\operatorname{Char} k = p > 0$. By the argument in the proof of Proposition 3.7(a), we can assume that S is finite. Then we put $G' = G \rtimes S$ and obtain, as in that proof, that S is G'-cr and that $C_G(S)$ is G-cr. Now the assumptions of Proposition 5.7(a) and (b) are satisfied. □

6. Centralizers and normalizers

In this section we continue the theme of Section 5, looking at the special case when S is a subgroup of G acting on G by inner automorphisms. The extra restriction on S allows us to consider subgroups H sitting between $C_G(S)^0$ and $N_G(S)$. The following result gives a criterion for K to be H-cr; it generalizes [4, Prop. 3.9] and the argument we use is very similar.

Proposition 6.1. *Suppose S is a G-completely reducible subgroup of G, and suppose H is a subgroup of G such that $C_G(S)^0 \subseteq H \subseteq N_G(S)$. Let K be a subgroup of H. Then:*

(a) *HS is reductive.*
(b) *The following are equivalent:*
  (i) *KS is G-completely reducible;*
  (ii) *KS is HS-completely reducible;*
  (iii) *KS is $N_G(S)$-completely reducible;*
  (iv) *KS/S is $N_G(S)/S$-completely reducible.*
(c) *Suppose that H is reductive. Let ψ be the canonical projection from H to $H/C_G(S)^0$. Then K is H-completely reducible if and only if KS/S is $N_G(S)/S$-completely reducible and $\psi(K)$ is $H/C_G(S)^0$-completely reducible.*

Proof. (a). We have $C_G(S)^0 S^0 = N_G(S)^0$, by [22, Lem. 6.8]. As $C_G(S)^0 \subseteq H$, we have $(HS)^0 = N_G(S)^0$. Now $N_G(S)$ is reductive by [3, Prop. 3.12], so HS is reductive.

(b). By part (a), $(HS)^0 = N_G(S)^0$, so HS is a finite-index subgroup of $N_G(S)$. The equivalence of (ii) and (iii) now follows from [4, Prop. 2.12]. The equivalence of (i) and (iii) follows from [4, Cor. 3.3], and the equivalence of (iii) and (iv) follows from [4, Thm. 3.4].

(c). Now suppose that H is reductive. The subgroups $H \cap S$ and $(H \cap S)^0$ are normal in H. Let π be the canonical projection from H to $H/(H \cap S)^0$. We have a homomorphism $\pi \times \psi$ from H to $H/(H \cap S)^0 \times H/C_G(S)^0$. If $h \in \ker(\pi \times \psi)$, then $h \in (H \cap S)^0 \cap C_G(S)^0 \subseteq S^0 \cap C_G(S)^0 \subseteq Z(S^0)$, so $\ker(\pi \times \psi)^0$ is a torus. By [3, Lem. 2.12(i) and (ii)], K is H-cr if and only if $\pi(K)$ is $H/(H \cap S)^0$-cr and $\psi(K)$ is $H/C_G(S)^0$-cr. To finish the proof, it is enough to show that $\pi(K)$ is $H/(H \cap S)^0$-cr if and only if KS/S is $N_G(S)/S$-cr. The natural map $H/(H \cap S)^0 \to HS/S$ is an isogeny, so $\pi(K)$ is $H/(H \cap S)^0$-cr if and only if KS/S is HS/S-cr, by [3, Lem. 2.12(ii)]. As HS/S is a finite-index subgroup of $N_G(S)/S$, KS/S is HS/S-cr if and only if KS/S is $N_G(S)/S$-cr [4, Prop. 2.12]. This completes the proof. □

Corollary 6.2. *Let G, H, S and K be as in Proposition* 6.1. *Suppose that $(H \cap S)^0$ is a torus. Then H is reductive and the following are equivalent:*

(i) *K is H-completely reducible;*
(ii) *KS is G-completely reducible;*
(iii) *KS is HS-completely reducible;*
(iv) *KS is $N_G(S)$-completely reducible;*
(v) *KS/S is $N_G(S)/S$-completely reducible.*

Proof. Since $N_G(S)^0 = C_G(S)^0 S^0$, and $C_G(S)^0 \subseteq H \subseteq N_G(S)$, we have $H^0 = C_G(S)^0 (H \cap S)^0$, so H is reductive. Moreover, $H^0/C_G(S)^0$ is a quotient of $(H \cap S)^0$, which is a torus by hypothesis. Hence $H/C_G(S)^0$ is a finite extension of a torus, which implies that any subgroup of $H/C_G(S)^0$ is $H/C_G(S)^0$-cr. Thus K is H-cr if and only if KS/S is $N_G(S)/S$-cr, by Proposition 6.1(c). The result now follows from Proposition 6.1(b). □

Remarks 6.3. (i). Suppose H is reductive. If K is H-cr, then Proposition 6.1(c) says that KS/S is $N_G(S)/S$-cr, whence KS is G-cr, by Proposition 6.1(b). The converse, however, is not true in this generality; just take $S = H = G$ and K a non-G-cr subgroup of G.

(ii). Corollary 6.2 holds in particular if S is linearly reductive, since then the condition that $(H \cap S)^0$ is a torus is automatic. However, Example 7.21 shows that even when S is linearly reductive, the situation for subgroups of $N_G(S)$ is not as straightforward as for subgroups of $C_G(S)$ (cf. Corollary 5.8).

7. An important example

We consider a collection of important examples, which serve to illustrate many of the points raised in the previous sections. Throughout this section, we suppose that $p = 2$ and let G be a simple group of type G_2. We fix a maximal torus T and a Borel subgroup B of G with $T \subseteq B$. Let Ψ be the set of roots of G with respect to T. We fix a base $\Sigma = \{\alpha, \beta\}$ for the set Ψ^+ of positive roots with respect to B, where α is short and β is long. The positive roots are α, β, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$ and $3\alpha+2\beta$. For each root γ, we choose an isomorphism $\kappa_\gamma \colon k \to U_\gamma$ and set $s_\gamma = \kappa_\gamma(1)\kappa_{-\gamma}(-1)\kappa_\gamma(1)$ (cf. [14, 32.3]). Then s_γ represents the reflection corresponding to γ in the Weyl group $N_G(T)/T$ of G. Since $p = 2$, the order of s_γ is 2 for every $\gamma \in \Psi$.

We use various equations from [14, 33.5], some of which are reproduced below. For brevity, we do not give the commutation relations between the root subgroups: these are the equations of the form

$$\kappa_\gamma(a)\kappa_{\gamma'}(b) = \kappa_{\gamma'}(b)\kappa_\gamma(a)g,$$

where g is a product of elements of the form $\kappa_{\gamma''}(p_{\gamma''}(a,b))$ over certain roots γ'', each $p_{\gamma''}$ being a monomial in a and b. (Recall, however, that U_γ and $U_{\gamma'}$ commute if no positive integral combination of γ and γ' is a root.) We refer to these equations collectively below as "the CRs".

Since G is simply connected, we have $G_\gamma \cong \mathrm{SL}_2(k)$ for every $\gamma \in \Psi$, by Lemma 2.2.

We have

$$\langle \alpha, \alpha^\vee \rangle = 2,\ \langle \beta, \alpha^\vee \rangle = -3,\ \langle \alpha, \beta^\vee \rangle = -1,\ \langle \beta, \beta^\vee \rangle = 2 \tag{7.1}$$

(see [14, 32.3]; note that $\langle \beta, \alpha \rangle$ in Humphreys's notation coincides with $\langle \beta, \alpha^\vee \rangle$). This yields

$$\langle \alpha+\beta, \alpha^\vee \rangle = -1,\ \langle 2\alpha+\beta, \alpha^\vee \rangle = 1,\ \langle 3\alpha+\beta, \alpha^\vee \rangle = 3,\ \langle 3\alpha+2\beta, \alpha^\vee \rangle = 0. \tag{7.2}$$

We have

$$s_\alpha \cdot \alpha = -\alpha,\ s_\alpha \cdot \beta = 3\alpha + \beta,$$

and it follows that

$$s_\alpha \cdot \alpha^\vee = -\alpha^\vee,\ s_\alpha \cdot \beta^\vee = \alpha^\vee + \beta^\vee. \tag{7.3}$$

We need to know how s_α acts on the U_γ. We can choose the homomorphisms κ_γ so that s_α maps each U_γ to $U_{s_\alpha \cdot \gamma}$ by conjugation in the following way (see [14, 33.1 and 33.5]):

(7.4)

$$s_\alpha\kappa_\beta(a)s_\alpha = \kappa_{3\alpha+\beta}(a),\ s_\alpha\kappa_{\alpha+\beta}(a)s_\alpha = \kappa_{2\alpha+\beta}(a),\ s_\alpha\kappa_{2\alpha+\beta}(a)s_\alpha = \kappa_{\alpha+\beta}(a),$$
$$s_\alpha\kappa_{3\alpha+\beta}(a)s_\alpha = \kappa_\beta(a),\ s_\alpha\kappa_{3\alpha+2\beta}(a)s_\alpha = \kappa_{3\alpha+2\beta}(a),$$

and we can choose $0 \neq e_\gamma \in \mathfrak{u}_\gamma$ for each positive root γ such that the adjoint action of s_α on the e_γ is given by

$$\begin{aligned} &\operatorname{Ad} s_\alpha(e_\beta) = e_{3\alpha+\beta},\ \operatorname{Ad} s_\alpha(e_{\alpha+\beta}) = e_{2\alpha+\beta},\ \operatorname{Ad} s_\alpha(e_{2\alpha+\beta}) = e_{\alpha+\beta},\\ &\operatorname{Ad} s_\alpha(e_{3\alpha+\beta}) = e_\beta,\ \operatorname{Ad} s_\alpha(e_{3\alpha+2\beta}) = e_{3\alpha+2\beta}. \end{aligned} \tag{7.5}$$

Using (7.4) and the CRs, we get

$$\begin{aligned} &s_\alpha \kappa_\beta(a)\kappa_{\alpha+\beta}(a')\kappa_{2\alpha+\beta}(b)\kappa_{3\alpha+\beta}(b')\kappa_{3\alpha+2\beta}(c)s_\alpha\\ =\ &\kappa_\beta(b')\kappa_{\alpha+\beta}(b)\kappa_{2\alpha+\beta}(a')\kappa_{3\alpha+\beta}(a)\kappa_{3\alpha+2\beta}(ab'+a'b+c). \end{aligned} \tag{7.6}$$

Let

$$L := \langle G_\alpha \cup T\rangle.$$

Let P be the parabolic subgroup of G that contains B and has L as a Levi subgroup. The roots of $R_u(P)$ with respect to T are β, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, $3\alpha+2\beta$.

Let S be the torus $\alpha^\vee(k^*)$; we have $S = G_\alpha \cap T$. Fix $t \in S$ such that t has order 3. By (7.1) and (7.2),

$$\begin{aligned} &t \text{ acts trivially on } U_\beta,\ U_{3\alpha+\beta},\ U_{3\alpha+2\beta},\ \mathfrak{u}_\beta,\ \mathfrak{u}_{3\alpha+\beta},\ \mathfrak{u}_{3\alpha+2\beta}, \text{ and}\\ &t \text{ acts non-trivially on } U_\alpha,\ U_{\alpha+\beta},\ U_{2\alpha+\beta},\ \mathfrak{u}_\alpha,\ \mathfrak{u}_{\alpha+\beta},\ \mathfrak{u}_{2\alpha+\beta}. \end{aligned} \tag{7.7}$$

Set

$$H := \langle\{s_\alpha, t\}\rangle \subseteq G_\alpha.$$

Note that $H \cong S_3$. Since $G_\alpha \cong \mathrm{SL}_2(k)$ (Lemma 2.2), $\alpha^\vee$ is an isomorphism from k^* onto $G_\alpha \cap T$. Set

$$z := d\alpha^\vee(1) \in \mathfrak{g}_\alpha = \operatorname{Lie} G_\alpha$$

(where we regard 1 as an element of the tangent space $T_1(k^*) \cong k$); then $z \neq 0$ and $k \cdot z$ is the centre of $\mathfrak{g}_\alpha$. In particular, H centralizes z. If $\gamma \in \Psi$, then

$$U_\gamma \text{ centralizes } z \iff 2 \text{ divides } \langle \gamma, \alpha^\vee\rangle. \tag{7.8}$$

By the CRs, G_α commutes with $G_{3\alpha+2\beta}$. The multiplication map $G_\alpha \times G_{3\alpha+2\beta} \to G_\alpha G_{3\alpha+2\beta}$ is bijective because $G_\alpha \cap G_{3\alpha+2\beta}$, being a proper normal subgroup of G_α, must be trivial, but this map is not an isomorphism. For $\mathfrak{z}(\mathfrak{g}_{3\alpha+2\beta}) \subseteq \operatorname{Lie}(G_{3\alpha+2\beta} \cap T)$, so we have $\mathfrak{z}(\mathfrak{g}_{3\alpha+2\beta}) \subseteq \mathfrak{c}_{\mathfrak{t}}(G_\alpha) \subseteq \mathfrak{c}_{\mathfrak{t}}(H)$, which equals $k \cdot z$, so $\mathfrak{g}_\alpha \cap \mathfrak{g}_{3\alpha+2\beta}$ is non-empty. In fact $\mathfrak{g}_\alpha \cap \mathfrak{g}_{3\alpha+2\beta} = k \cdot z$, because $k \cdot z$ is the unique proper non-zero normal subalgebra of $\operatorname{Lie} G_\alpha = \mathfrak{sl}_2(k)$.

Now set

$$M := G_\alpha G_{3\alpha+2\beta}.$$

Note that $T \subseteq M$ and M is a semisimple maximal rank subgroup of G of type $\widetilde{A}_1 A_1$. Consequently, M is G-ir. Moreover, we have

$$C_G(M) = \ker(\alpha) \cap \ker(3\alpha+2\beta) = \ker(\alpha)\cap\ker(2\beta) = \ker(\alpha)\cap\ker(\beta) = Z(G) = \{1\}.$$

It is easy to see that $\mathfrak{z}(\mathfrak{m}) = k \cdot z$.

Lemma 7.9. *With the notation as above, we have*

(a) (G, M) *is a reductive pair;*
(b) $N_G(M) = M$*;*
(c) $M = C_G(z)$.

Proof. (a). Since $\Psi(M)$ is a closed subsystem of Ψ, this follows immediately from Lemma 3.9.

(b). Let $g \in N_G(M)$. Without loss of generality, we can assume that g normalizes T, so g permutes $\Psi(M) = \{\pm\alpha, \pm(3\alpha+2\beta)\}$. As α is short and $3\alpha+2\beta$ is long, g must normalize G_α and $G_{3\alpha+2\beta}$. But G_α and $G_{3\alpha+2\beta}$ are rank one groups, so they have no outer automorphisms. Since $C_G(M) = \{1\}$, we therefore have that $g \in M$, as required.

(c). Since $C_G(z)^0 \supseteq M$ and M is G-ir, $C_G(z)^0$ is G-ir, so $C_G(z)^0$ is reductive. This implies that $C_G(z)^0$ is generated by the root subgroups that it contains together with T. It follows from (7.1), (7.2) and (7.8) that $C_G(z)^0 = M$. Hence $C_G(z) \subseteq N_G(C_G(z)^0) = N_G(M) = M$, by part (b). Thus $C_G(z) = M$. □

Lemma 7.10. *With the notation as above, we have*

(a) *H is G-completely reducible and M-completely reducible;*
(b) *$C_G(H) = G_{3\alpha+2\beta}$ and $N_G(H) = HG_{3\alpha+2\beta}$.*

Proof. (a). It is easily checked that H is not contained in any Borel subgroup of L, so H is L-ir. Now L is a Levi subgroup both of G and of M, so H is both G-cr and M-cr by [3, Cor. 3.22].

(b). Since H is G-cr, $C_G(H)$ is G-cr by [3, Cor. 3.17], so $C_G(H)^0$ is reductive. Now $C_G(H)^0$ cannot have rank 2, because H is not centralized by any maximal torus of G. Hence $C_G(H)^0$ must be equal to its rank 1 subgroup $G_{3\alpha+2\beta}$. Now $G_{3\alpha+2\beta} = C_G(H)^0$ is G-cr by [3, Thm. 3.10], so $C_G(G_{3\alpha+2\beta})^0 = G_\alpha$ by a similar argument to that for $C_G(H)^0$.

Let $g \in N_G(H)$. Then g normalizes $C_G(H)^0 = G_{3\alpha+2\beta}$, so g normalizes $C_G(G_{3\alpha+2\beta})^0 = G_\alpha$, so g normalizes M. Lemma 7.9(b) now implies that $N_G(H) = N_M(H) = HG_{3\alpha+2\beta}$ and $C_G(H) = C_M(H) = G_{3\alpha+2\beta}$. □

The following example shows that the converse of Corollary 3.8 can fail, even when H is L-ir. It also gives a counterexample to the converse of the first assertion of Theorem 1.4 (note that (G, L) is a reductive pair by Lemma 3.9).

Proposition 7.11. *The subgroup H is separable in L, but not in G.*

Proof. Since $DL = G_\alpha \cong \mathrm{SL}_2(k)$, L is isomorphic either to $\mathrm{GL}_2(k)$ or to $\mathrm{SL}_2(k) \times k^*$. To rule out the latter case, it's enough to show that s_α acts non-trivially on $\mathfrak{t}$; this follows from (7.3). It now follows easily that $\mathfrak{c}_{\mathfrak{l}}(H) = k \cdot z$. Now z is tangent to $\mathrm{Lie}\, Z(L)$ as $L \cong \mathrm{GL}_2(k)$. We deduce that H is separable in L.

It follows from (7.5) and (7.7) that H centralizes $e_\beta + e_{3\alpha+\beta}$. But $C_G(H) = G_{3\alpha+2\beta}$, by Lemma 7.10(b), so $e_\beta + e_{3\alpha+\beta}$ is not tangent to $\mathrm{Lie}\, C_G(H) = \mathfrak{g}_{3\alpha+2\beta}$. Hence H is not separable in G. □

Remark 7.12. It is easily checked that for every semisimple $x \in \mathfrak{c}_{\mathfrak{g}}(H)$, we have $x \in \mathrm{Lie}\, C_G(H)$. The same result cannot hold if we replace H with a G-irreducible and non-separable subgroup of G; cf. the proof of [3, Thm. 3.39].

For $a \in k$, set

$$u(a) = \kappa_\beta(a)\kappa_{3\alpha+\beta}(a).$$

The CRs yield

$$u(a)u(b) = u(a+b)\kappa_{3\alpha+2\beta}(ab) \tag{7.13}$$

and

$$u(a)^{-1} = u(a)\kappa_{3\alpha+2\beta}(a^2). \tag{7.14}$$

Define

$$H_a = u(a)Hu(a)^{-1}.$$

By (7.7), $u(a)$ centralizes t; by (7.6), we have $u(a)s_\alpha u(a)^{-1} = s_\alpha \kappa_{3\alpha+2\beta}(a^2)$. Hence $H_a = \langle\{t, s_\alpha\kappa_{3\alpha+2\beta}(a^2)\}\rangle \subseteq M$ for all $a \in k$.

The following example shows that Theorem 1.3(b) can fail if we do not require K to be separable in G (recall that (G, M) is a reductive pair by Lemma 7.9(a)). This example also proves Theorem 1.5.

Example 7.15. Let $(m_1, m_2) = (s_\alpha, t\kappa_{3\alpha+2\beta}(1))$. For $a \in k$, set

$$(m_1(a), m_2(a)) = u(a) \cdot (m_1, m_2).$$

By construction, $(m_1(a), m_2(a))$ is G-conjugate to $(m_1(b), m_2(b))$ for all $a, b \in k$. We now show that if $a \neq b$, then $(m_1(a), m_2(a))$ is not M-conjugate to $(m_1(b), m_2(b))$. This shows that $G \cdot (m_1, m_2)$ is an infinite union of M-conjugacy classes.

For $a \in k$ let

$$\widehat{H}_a := \langle\{m_1(a), m_2(a)\}\rangle.$$

Note that t commutes with $\kappa_{3\alpha+2\beta}(1)$ by (7.7), so

$$\widehat{H}_0 = \langle\{m_1, m_2\}\rangle = \langle\{s_\alpha, t, \kappa_{3\alpha+2\beta}(1)\}\rangle = \langle H \cup \{\kappa_{3\alpha+2\beta}(1)\}\rangle$$

and $\widehat{H}_a = \langle H_a \cup \{\kappa_{3\alpha+2\beta}(1)\}\rangle$ similarly. Hence

$$C_G(\widehat{H}_0) = C_G(H) \cap C_G(\kappa_{3\alpha+2\beta}(1)) = G_{3\alpha+2\beta} \cap C_G(\kappa_{3\alpha+2\beta}(1)) = U_{3\alpha+2\beta},$$

by Lemma 7.10(b), so

$$\begin{aligned} C_G(\widehat{H}_a) = C_G(u(a)\widehat{H}_0 u(a)^{-1}) &= u(a)C_G(\widehat{H}_0)u(a)^{-1} \\ &= u(a)U_{3\alpha+2\beta}u(a)^{-1} = U_{3\alpha+2\beta}. \end{aligned}$$

Now let $a, b \in k$, and suppose that $(m_1(a), m_2(a))$ and $(m_1(b), m_2(b))$ are M-conjugate. Then there exists $m \in M$ such that $(mu(a)) \cdot (m_1, m_2) = u(b) \cdot (m_1, m_2)$. We have $mu(a)u(b)^{-1} \in C_G(\langle\{m_1(b), m_2(b)\}\rangle) = C_G(\widehat{H}_b) = U_{3\alpha+2\beta} \subseteq M$, so $u(a)u(b)^{-1} \in M$. But $u(a)u(b)^{-1} = u(a+b)\kappa_{3\alpha+2\beta}(ab + b^2)$ by (7.13) and (7.14), so $u(a+b) \in M$. So we must have $u(a+b) = 1$, whence $a = b$. Thus if $a \neq b$, then $(m_1(a), m_2(a))$ and $(m_1(b), m_2(b))$ are not M-conjugate.

These calculations show that even though (G, M) is a reductive pair, $G \cdot (m_1, m_2) \cap M^2$ consists of an infinite union of M-conjugacy classes. Observe that this is consistent with Theorem 1.3(b); a similar calculation to the one in the proof of Proposition 7.11 shows that $\widehat{H}_0$ is not separable in G.

Remark 7.16. The n-tuple $(m_1, m_2, \dots, m_n)$ yields a similar example for any $n \geq 2$, where m_1 and m_2 are as above and $m_3 = \cdots = m_n = 1$.

Our next example shows that the second assertion of Theorem 1.4 is false with the separability assumption on K removed, even though (G, M) is a reductive pair, by Lemma 7.9(a). Since M is a regular subgroup of G, this is also a new example of the failure of [3, Thm. 3.26] in bad characteristic (compare [3, Ex. 3.45], which gives subgroups $H' \subseteq M' \subseteq G'$, with G' connected reductive, M' regular in G' such that H' is M'-cr, but not G'-cr).

Proposition 7.17. *Let $a \in k^*$. Then H_a is G-completely reducible but not M-completely reducible.*

Proof. Since H_a is G-conjugate to H and H is G-cr (Lemma 7.10(a)), H_a is G-cr. Let $\lambda = \alpha^\vee + 2\beta^\vee \in Y(T)$; then $\langle \alpha, \lambda \rangle = 0$ and $\langle \beta, \lambda \rangle = 1$. It is clear that $P = P_\lambda$ and $L = L_\lambda$. We have a homomorphism $c_\lambda \colon P_\lambda \to L_\lambda$ as defined in Subsection 2.2. If $h \in H$, then $u(a)hu(a)^{-1} = hu$ for some $u \in U_{3\alpha+2\beta} \subseteq \ker(c_\lambda)$, so $c_\lambda(H_a) = H$. To prove that H_a is not M-cr, it suffices by [3, Lem. 2.17 and Thm. 3.1] to show that H_a and H are not M-conjugate. But this is the case, since $u(a) \notin M$ and $N_G(H) \subseteq M$ (Lemma 7.10(b)), so we are done. □

Remark 7.18. Proposition 7.17 shows that part (c) of Theorem 1.3 fails without the separability hypothesis on K, for, by Theorem 2.9, the G-conjugacy class $G \cdot (m_1(a), m_2(a))$ is closed in $G \times G$ but $M \cdot (m_1(a), m_2(a))$ is not.

The next example shows that Proposition 2.14 can fail if we allow H to be non-G-cr. First we need a refinement of Lemma 3.1. If G_1 is a reductive group and H_1 is a subgroup of G_1, then we say that $x \in \mathfrak{g}_1$ is a *witness to the non-separability of* H_1 if $x \in \mathfrak{c}_{\mathfrak{g}_1}(H_1)$ but $x \notin \operatorname{Lie} C_{G_1}(H_1)$.

Lemma 7.19. *Let $f \colon G_1 \to G_2$ be an epimorphism of reductive groups such that $\ker df$ consists of semisimple elements and let H_1 be a subgroup of G_1. Let $x \in \mathfrak{c}_{\mathfrak{g}_1}(H_1)$ be nilpotent. Then x is a witness to the non-separability of H_1 if and only if $df(x)$ is a witness to the non-separability of $f(H_1)$.*

Proof. It is clear that $df(x) \in \mathfrak{c}_{\mathfrak{g}_2}(f(H_1))$ and that $df(x)$ is tangent to $C_{G_2}(f(H_1))$ if x is tangent to $C_{G_1}(H_1)$. Conversely, suppose that $df(x)$ is tangent to $C_{G_2}(f(H_1))$. By [7, Prop. IV.14.26], there exists a connected unipotent subgroup U_2 of $C_{G_2}(f(H_1))$ such that $df(x) \in \operatorname{Lie} U_2$. By [7, V.22.1], the restriction of f is an isogeny from $D(G_1^0)$ to $D(G_2^0)$. Let $U_1 = (f^{-1}(U_2) \cap D(G_1^0))^0$. Any semisimple element of U_1 must belong to the finite group $\ker f \cap D(G_1^0)$, so U_1 has only finitely many semisimple elements. Hence U_1 is a unipotent group. As $\ker df$ consists of semisimple elements and $U_2 \subseteq D(G_2^0)$, it follows that the restriction of f is an isomorphism from U_1 onto U_2. Hence we can choose $x' \in \operatorname{Lie} U_1$ such that $df(x') = df(x)$. Now $\ker df \subseteq \mathfrak{z}(\mathfrak{g}_1)$, by [7, V.22.2], and x, x' are both nilpotent, so we must have $x = x'$, whence $x \in \operatorname{Lie} U_1$.

To complete the proof, it suffices to show that $U_1 \subseteq C_{G_1}(H_1)$. Fix $h \in H_1$. If $u \in U_1$, then $f(h)f(u)f(h)^{-1} = f(u)$, so we have $huh^{-1} = cu$ for some $c \in \ker f$. The map $u \mapsto huh^{-1}u^{-1}$ is therefore a morphism from the connected set U_1 to the finite set $\ker f \cap D(G_1^0)$, so $huh^{-1}u^{-1} = 1$ for all $h \in H_1$ and all $u \in U_1$. Hence $U_1 \subseteq C_{G_1}(H_1)$, as required. □

Example 7.20. Let $C = \{u(a) \mid a \in k\}$. Let $H' = \langle H \cup C \rangle$ and suppose that K is any reductive subgroup of G containing H'. Set $y := e_\beta + e_{3\alpha+\beta}$. Then y belongs to the tangent space $T_1(C)$, which is contained in $\operatorname{Lie} K$. Now $\mathfrak{u}_{3\alpha+2\beta}$ is centralized by U_β and $U_{3\alpha+2\beta}$ and we have $\operatorname{Ad} \kappa_{3\alpha+\beta}(a)(e_\beta) = e_\beta + ae_{3\alpha+2\beta}$, $\operatorname{Ad} \kappa_\beta(a)(e_{3\alpha+\beta}) = e_{3\alpha+\beta} + ae_{3\alpha+2\beta}$, by the CRs, so $\operatorname{Ad} u(a)(y) = y$. Hence $y \in \mathfrak{c}_{\mathfrak{k}}(H')$, but y is not tangent to $C_K(H')$, since it is not tangent to $C_G(H)$ (cf. the proof of Proposition 7.11). By Lemma 7.19, y is a witness to the non-separability of $\pi_K(H')$, so $\pi_K(H')$ is not separable in K_{ad}.

Here is a further example arising from this construction which relates to the discussions in Sections 3 and 6 (see in particular Remarks 3.10(ii) and 6.3(ii)).

Example 7.21. Recall that S is the torus $\alpha^\vee(k^*)$ and therefore, $(G, N_G(S))$ is a reductive pair, by Proposition 3.7(b). We have that $H_a \subseteq N_G(S)$, since $U_{3\alpha+2\beta}$ centralizes S. Let $a \in k^*$. We will show that H_aS is not G-cr, although H_a is (Proposition 7.17). Let $\lambda \in Y(T)$ and let $c_\lambda : P_\lambda \to L_\lambda$ be as defined in Subsection 2.2. Then

$$c_\lambda(H_aS) = \langle S \cup \{s_\alpha\}\rangle.$$

If H_aS lies in a Levi subgroup of P_λ, then $uH_aSu^{-1} \subseteq L_\lambda$ for some $u \in R_u(P_\lambda)$, so $c_\lambda(H_aS) = uH_aSu^{-1}$. Thus to show that H_aS is not G-cr, it is enough to show that $\langle S \cup \{s_\alpha\}\rangle$ is not $R_u(P_\lambda)$-conjugate to H_aS. To see this, note that if $u \in R_u(P_\lambda)$ with $u\langle S \cup \{s_\alpha\}\rangle u^{-1} = H_aS$, then $uSu^{-1} = S$, whence u centralizes S (since S normalizes $R_u(P_\lambda)$). Since the centralizer of S in $R_u(P_\lambda)$ is $U_{3\alpha+2\beta}$, by (7.1) and (7.2), we have $u \in U_{3\alpha+2\beta}$. But $U_{3\alpha+2\beta}$ centralizes H_aS, so $\langle S \cup \{s_\alpha\}\rangle = H_aS$, a contradiction.

Since H_aS is not G-cr, H_aS is not $N_G(S)$-cr (Corollary 6.2). Since the canonical projection $f\colon N_G(S) \to N_G(S)/S$ is non-degenerate and $f(H_a) = f(H_aS)$, it follows from [3, Lem. 2.12(ii)] that H_a is not $N_G(S)$-cr. Thus we have an example of a subgroup $H_a \subseteq N_G(S)$, with S linearly reductive — in fact, a torus — such that H_a is G-cr, but not $N_G(S)$-cr.

Finally, we consider a rationality question. Let k_0 be a subfield of an algebraically closed field k_1. Suppose that G_1 is a reductive algebraic group defined over k_0. If K_1 is a subgroup of G_1 defined over k_0, then we say that K_1 is G_1-*completely reducible over* k_0 if whenever P_1 is an R-parabolic subgroup of G_1 such that $K_1 \subseteq P_1$ and P_1 is defined over k_0, there exists an R-Levi subgroup L_1 of P_1 such that $K_1 \subseteq L_1$ and L_1 is defined over k_0 (see [3, Sec. 5] for further details). In particular, K_1 is G_1-cr if and only if K_1 is G_1-cr over k_1. An example of McNinch [3, Ex. 5.11] shows that if K_1 is G_1-cr over k_0, then K_1 need not be G_1-cr over k_1. The next example shows that the converse can also happen.

Example 7.22. Suppose that k_0 is a subfield of k such that G is defined over k_0 and k_0-split. We can assume that T is chosen so that T is defined over k_0 and k_0-split and so that for every $\gamma \in \Psi$, the homomorphisms $\gamma\colon T \to k^*$, $\gamma^\vee\colon k^* \to T$ and $\kappa_\gamma\colon k \to U_\gamma$ are defined over k_0. Now suppose that k/k_0 is not separable. Then k_0 is not perfect, so there exists $a \in k \setminus k_0$ such that $a^2 \in k_0$. Consider $H_a = \langle\{t, s_\alpha\kappa_{3\alpha+2\beta}(a^2)\}\rangle$. Then H_a is defined over k_0, since H_a is a finite subgroup of $G(k_0)$. As H_a is G-cr (Proposition 7.17), H_a is G-cr over k. We show that H_a is not G-cr over k_0.

Recall that $P = P_\lambda$ and $L = L_\lambda$, where λ is as in the proof of Proposition 7.17. Since G and T are split, P and L are defined over k_0 (cf. [7, Props. V.20.4 and V.20.5]). Suppose there exists a Levi subgroup L' of P such that L' is defined over k_0 and H_a is contained in L'. By [7, Prop. V.20.5], there exists $u \in R_u(P)(k_0)$ such that $L' = uLu^{-1}$. Then $u^{-1}H_au \subseteq L$, and $u(a)^{-1}H_au(a) \subseteq L$, so $c_\lambda(u^{-1}gu) = u^{-1}gu$ and $c_\lambda(u(a)^{-1}gu(a)) = u(a)^{-1}gu(a)$ for all $g \in H_a$. Since $c_\lambda(u) = c_\lambda(u(a)) = 1$, we have

$$u^{-1}gu = c_\lambda(u^{-1}gu) = c_\lambda(g) = c_\lambda(u(a)^{-1}gu(a)) = u(a)^{-1}gu(a)$$

for all $g \in H_a$. Thus $u = u(a)c$ for some $c \in C_G(H) \cap R_u(P) = G_{3\alpha+2\beta} \cap R_u(P) = U_{3\alpha+2\beta}$ (using Lemma 7.10(b)): say $u = \kappa_\beta(a)\kappa_{3\alpha+\beta}(a)\kappa_{3\alpha+2\beta}(y)$ for some $y \in k$. But then $u \notin P(k_0)$, since $a \notin k_0$, a contradiction. Thus no such L' can exist, and H_a is not G-cr over k_0.

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