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The Brauer algebra and the symplectic Schur algebra

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Abstract Let k be an algebraically closed field of characteristic $p > 0$, let m, r be integers with $m \geq 1$, $r \geq 0$ and $m \geq r$ and let $S_0(2m, r)$ be the symplectic Schur algebra over k as introduced by the first author. We introduce the symplectic Schur functor, derive some basic properties of it and relate this to work of Hartmann and Paget. We do the same for the orthogonal Schur algebra. We give a modified Jantzen sum formula and a block result for the symplectic Schur algebra under the assumption that r and the residue of $2m$ mod p are small relative to p. From this we deduce a block result for the orthogonal Schur algebra under similar assumptions. We also show that, in general, the block relations of the Brauer algebra and the symplectic and orthogonal Schur algebra are the same. Finally, we deduce from the previous results a new proof of the description of the blocks of the Brauer algebra in characteristic 0 as obtained by Cox, De Visscher and Martin.

Keywords Brauer algebra · Symplectic Schur algebra · Jantzen sum formula · Young modules · Blocks

Mathematics Subject Classification (2000) 14L35 · 05E15

Introduction

In characteristic zero the strong relationship between the representation theories of the general linear group and the symmetric group, is well-known; see, e.g. [41]. In Green's monograph [25] a characteristic free approach to this is given, using the Schur functor as defined by Schur in his doctoral dissertation. In this work we give a systematic Lie theoretic approach to the representation theory of the Brauer algebra in the spirit of Green's monograph. Our

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approach was stimulated by a result of Cox et al. [10] expressing the blocks of the Brauer algebra in characteristic zero in terms of Weyl group orbits, and a desire to see this result in a Lie theoretic context. Throughout the paper we work over an algebraically closed field k.

The paper is organized as follows. In Sect. 1 we introduce the necessary notation, including the Brauer algebra $B_r = B_r(\delta)$ and for $n = 2m$ even, the symplectic group Sp_n and the symplectic Schur algebra $S_0(n, r)$. Furthermore, we introduce Specht, permutation and Young modules for the Brauer algebra as in [27] and their twisted versions.

In Sect. 2 we introduce the symplectic Schur functor

$$f_0 : \mathrm{mod}(S_0(n, r)) \to \mathrm{mod}(B_r(-n))$$

and the inverse symplectic Schur functor

$$g_0 : \mathrm{mod}(B_r(-n)) \to \mathrm{mod}(S_0(n, r)).$$

We assume that $m \geq r$ to ensure that the Brauer algebra $B_r(-n)$ can be identified with the endomorphism algebra $\mathrm{End}_{\mathrm{Sp}_n}(E^{\otimes r})$. Only in this situation can we expect the symplectic Schur functor to "control" the representation theory of the Brauer algebra. The main results are Theorem 2.1 and Propositions 2.1 and 2.2. These results express the link between the representation theories of the symplectic Schur algebra and the Brauer algebra. They will be needed for the results in Sect. 5.

In Sect. 3 we study the symplectic Schur algebra in the situation that $\mathrm{char}\, k = p > 2$ and r and the residue of n mod p are small relative to p. This section is independent of Sect. 2. We obtain a description of the blocks, Theorem 3.2, which is the same as the description of the blocks of the Brauer algebra in characteristic zero in [10]. Our main tool is a strengthened version of the Jantzen Sum Formula, Theorem 3.1.

In Sect. 4 we obtain the orthogonal versions of the results in Sect. 2. We assume here that $\mathrm{char}\, k \neq 2$. Since we are assuming that $n > 2r$, we can work with the special orthogonal group and avoid working with the full orthogonal group. Mostly the arguments are the same as in the symplectic case, and in that case they are omitted. The results in this section are important since they allow us to pass to the (untwisted) Specht, permutation and Young modules for the Brauer algebra via a Schur functor.

In Sect. 5, we use the results of the previous three sections to obtain block results. First we assume that $\mathrm{char}\, k = p > 2$ and show that the block relations of the symplectic and orthogonal Schur algebras are the same as those of the corresponding Brauer algebras, Theorem 5.1. From this we deduce generic block results for the Brauer algebra and the orthogonal Schur algebra. In Sect. 5.3 we assume that $\mathrm{char}\, k = 0$ and deduce the description of the blocks of the Brauer algebra [10, Thm. 4.2]. For this we use reduction mod p. The key point about working in positive characteristic p is that we can then subtract multiples of p from δ to get that $\delta - up = -n = -2m, m \geq r$, without changing the Brauer algebra: $B_r(\delta) = B_r(-n)$. In the situation that $m \geq r$ we can then exploit the relation between the representation theories of the symplectic Schur algebra $S_0(n, r)$ and the Brauer algebra $B_r(-n)$. For the reduction mod p to work we need that, for a fixed integer δ, the blocks of the Brauer algebra over a field of characteristic zero "agree" with the blocks over a field of large prime characteristic. This is a very general fact, as is explained in Sect. 5.2. The idea that characteristic zero theory is the limiting case of characteristic p theory has also been used for the partition algebra in [33].

Throughout this paper we will freely make use of the general theory of quasihereditary algebras, the theory of reductive groups and their representations and the representation theory of the symmetric group. For quasihereditary algebras (e.g. (co)standard module, ∇-filtration) we refer to [19, Appendix]. For reductive groups and their representations (e.g.

induced module, Weyl module, good filtration) we refer to [29, Part II]. Note that in [29] the induced and Weyl module with highest weight λ for a reductive group are denoted by $H^0(\lambda)$ and $V(\lambda)$, respectively. For the representation theory of the symmetric group (e.g. Specht module, permutation module, p-regular partition) we refer to [28]. A definition of Young modules for the symmetric group can be found in [34].

1 Preliminaries

1.1 The Brauer algebra and the symplectic Schur algebra

Throughout the paper k denotes an algebraically closed field. Let $n = 2m$ be an even integer ≥ 2. Let $i \mapsto i'$ be the involution of $\{1, \ldots, n\}$ defined by $i' := n+1-i$. Set $\epsilon_i = 1$ if $i \leq m$ and $\epsilon_i = -1$ if $i > m$ and define the $n \times n$-matrix J with coefficients in k by $J_{ij} = \delta_{ij'}\epsilon_i$. So

$$J = \begin{bmatrix} & & & & & 1 \\ & 0 & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & 0 & \\ -1 & & & & & \end{bmatrix}.$$

Let $E = k^n$ be the space of column vectors of length n with standard basis $e_1, \ldots, e_n$. On E we define the nondegenerate symplectic form $\langle\ ,\ \rangle$ by

$$\langle u, v\rangle := u^T J v = \sum_{i=1}^{n} \epsilon_i u_i v_{i'}.$$

Then $\langle e_i, e_j\rangle = J_{ij}$. The *symplectic group* $\mathrm{Sp}_n = \mathrm{Sp}_n(k)$ is defined as the group of $n \times n$-matrices over k that satisfy $A^T J A = J$, i.e. the invertible matrices for which the corresponding automorphism of E preserves the form $\langle\ ,\ \rangle$. We denote the general linear group over k by GL_n or $\mathrm{GL}_n(k)$. The vector space E is the natural module for GL_n and for Sp_n.

Let r be an integer ≥ 0. For any $\delta \in k$ one has the Brauer algebra $B_r(\delta)$; see, e.g. [4,5,21,40] or [9] for a definition. This also makes sense for δ an integer, since we can replace that integer by its natural image in k. Let $E^{\otimes r}$ be the r-fold tensor power of E. Then we have natural homomorphisms $k\mathrm{Sym}_r \to \mathrm{End}_{\mathrm{GL}_n}(E^{\otimes r})$ and $B_r(-n) \to \mathrm{End}_{\mathrm{Sp}_n}(E^{\otimes r})$. The action of the symmetric group Sym_r is by permutation of the factors, the action of $B_r(-n)$ is explained in [40, p. 192]. Using classical invariant theory one can then show that these homomorphisms are surjective and that they are injective in case $n \geq r$ and $m \geq r$, respectively; see [11,39] (the proof of the surjectivity is Brauer's original argument [4]). In the case of the symplectic group these results were first obtained in arbitrary characteristic in [12] using different methods. Let $S(n,r)$ and $S_0(n,r)$ be the enveloping algebras in $\mathrm{End}(E^{\otimes r})$ of GL_n and Sp_n, respectively. Then the natural embeddings $S(n,r) \to \mathrm{End}_{k\mathrm{Sym}_r}(E^{\otimes r})$ and $S_0(n,r) \to \mathrm{End}_{B_r(-n)}(E^{\otimes r})$ are isomorphisms; see [25,39]. In the case of the symplectic group this result was first obtained in arbitrary characteristic by Oehms [35]. The proof given in [39] avoids the FRT-construction, but makes essential use of the bideterminant basis given in [35, Thm. 6.1]. The algebra $S(n,r)$ is the *Schur algebra* as introduced in [25] and we will call $S_0(n,r)$ the *symplectic Schur algebra*, it was first introduced in [15].

In [13] *generalized Schur algebras* were introduced. These are quasi-hereditary finite dimensional algebras that are associated to a reductive group and a finite saturated set of weights. Let T and T_0 be the maximal tori of GL_n and Sp_n, respectively, that consist of diagonal matrices and let B and B_0 be the Borel subgroups that consist of upper triangular matrices. Note that T_0 consists of those $t \in T$ which satisfy $t_i t_{i'} = 1$ for all $i \in \{1, \dots, m\}$. Associated to a maximal torus and a Borel subgroup containing it one has a root datum and a choice of positive roots. We call the characters (multiplicative one-parameter subgroups) of a fixed maximal torus of a reductive group *weights*. Recall that a weight λ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for every positive root α. We denote the set of weights of GL_n with respect to T by X and the set of weights of Sp_n with respect to T_0 by X_0. The groups X and X_0 can be identified with $\mathbb{Z}^n$ and $\mathbb{Z}^m$, respectively.

For $l < n$ we identify $\mathbb{Z}^l$ with the sublattice of $\mathbb{Z}^n$ that consists of the n-tuples with the last $n - l$ components equal to 0.

For $i \in \{1, \dots, n\}$ we denote the element of $\mathbb{Z}^n$ which is 1 on the ith position and 0 elsewhere by ε_i. The character corresponding to ε_i is for GL_n, and also for Sp_n if $i \leq m$, the ith diagonal entry function. The map $\lambda \mapsto \overline{\lambda} : \mathbb{Z}^n \to \mathbb{Z}^m$ corresponding to restriction of characters sends ε_i to ε_i if $i \leq m$ and to $-\varepsilon_{i'}$ if $i > m$.

Recall that a function on a closed subgroup of GL_n is called *polynomial* if it is the restriction of a function on GL_n which is a polynomial in the matrix entries. Clearly all regular functions on Sp_n and T_0 are polynomial. The set of polynomial weights of GL_n with respect to T corresponds, under the above identifications, to the subset $\mathbb{N}^n$ of $\mathbb{Z}^n$ that consists of the compositions $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$ of some integer $|\lambda| := \sum_i \lambda_i$ into at most n parts. The set of dominant polynomial weights with respect to T and B corresponds to the subset $\Lambda^+(n)$ of $\mathbb{Z}^n$ that consists of the partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ of some integer into at most n parts. The set of dominant weights of Sp_n with respect to T_0 and B_0 corresponds to the set $\Lambda_0^+(m) = \Lambda^+(m) \subseteq \mathbb{Z}^m$. Define the sets of weights $\Lambda(n, r)$, $\Lambda^+(n, r)$, $\Lambda_0(m, r)$ and $\Lambda_0^+(m, r)$ by

$$\Lambda(n, r) := \{\lambda \in \mathbb{N}^n \mid |\lambda| = r\}, \ \Lambda^+(n, r) = \Lambda(n, r) \cap \Lambda^+(n),$$
$$\Lambda_0(m, r) := \{\lambda \in \mathbb{Z}^m \mid \textstyle\sum_i |\lambda_i| \leq r, r - |\lambda| \text{ even}\}, \quad \text{and}$$
$$\Lambda_0^+(m, r) := \Lambda_0(m, r) \cap \Lambda^+(m) = \{\lambda \in \Lambda^+(m) \mid |\lambda| \leq r, r - |\lambda| \text{ even}\}.$$

Note that, under the above identifications, $\Lambda^+(n, r) = \Lambda^+(r, r)$ if $n \geq r$ and that $\Lambda_0^+(m, r) = \Lambda_0^+(r, r)$ if $m \geq r$. The algebra $S(n, r)$ is the generalized Schur algebra associated to GL_n and $\Lambda^+(n, r)$ by [20, Thm 8.3]. Furthermore, one can deduce in the same way that $S_0(n, r)$ is the generalized Schur algebra associated to Sp_n and $\Lambda_0^+(m, r)$. Here we prefer to work with the symplectic group rather than the symplectic similitude group as in [15]. For $S(n, r)$ we denote the standard, costandard and irreducible module associated to $\lambda \in \Lambda^+(n, r)$ by $\Delta(\lambda)$, $\nabla(\lambda)$ and $L(\lambda)$, respectively. Occasionally, we will also use this notation for an arbitrary quasi-hereditary algebra or an arbitrary connected reductive group. For $S_0(n, r)$ and $\lambda \in \Lambda_0^+(m, r)$ we denote these modules by $\Delta_0(\lambda)$, $\nabla_0(\lambda)$ and $L_0(\lambda)$. In case of $S(n, r)$, $S_0(n, r)$ or a connected reductive group these are the Weyl, induced and irreducible module associated to λ for the group. Here we induce from the opposite Borel subgroups of B and B_0 to GL_n and Sp_n.

Later on we will need the following lemma which is, no doubt, well-known. Except for the second assertion of (i) (this can be found in [3, Thm. 2.1] and [38], for example), we couldn't find the result in the literature, so we include a proof.

Lemma 1.1

(i) *Let M be a finite dimensional vector space over k. The $k\mathrm{GL}(M)$-module M is a direct summand of $M \otimes M^* \otimes M$ and if $\dim M \neq 0$ in k, then the trivial $k\mathrm{GL}(M)$-module k is a direct summand of $M \otimes M^*$.*

(ii) *Let G be a group and let M be a self-dual finite dimensional kG-module. Let r and t be integers with $0 \leq t \leq r$ and $r - t$ even. Then $M^{\otimes t}$ is a direct summand of $M^{\otimes r}$ if $t \geq 1$ or $\dim M \neq 0$ in k.*

Proof (i) Put $l = \dim M$, let $(v_1, \dots, v_l)$ be a basis of M and let $(v_1^*, \dots, v_l^*)$ be the dual basis of M^*. Let $a : k \to M \otimes M^*$ be the map $\alpha \mapsto \alpha \sum_{i=1}^{l} v_i \otimes v_i^*$, let $b : M \to M \otimes M^* \otimes M$ be the map $x \mapsto x \otimes \sum_{i=1}^{l} v_i^* \otimes v_i$ and let $c : M \otimes M^* \to k$ be the contraction by means of the canonical bilinear form. Then one easily checks that $(c \otimes \mathrm{id}) \circ b = \mathrm{id}$ and that $c \circ a = l\,\mathrm{id}$. This proves (i).

(ii) By (i) we have that M is a direct summand of $M^{\otimes 3}$ and, if $\dim M \neq 0$ in k, the trivial kG-module k is a direct summand of $M^{\otimes 2}$. The assertion now follows by induction. □

1.2 Modules for the Brauer algebra

Notation *In what follows, s and t are not necessarily fixed integers ≥ 0 such that $r = t + 2s$.*

Let $\delta \in k$. For any integer $i \geq 0$, let $I_{s,i}$ be the left ideal of the Brauer algebra $B_r = B_r(\delta)$ spanned by the diagrams of which the bottom row has s horizontal edges which each join two consecutive nodes of the last $2s$ nodes and has at least i other horizontal edges. Put $I_s := I_{s,0}$, $Z_{s,i} := I_{s,i}/I_{s,i+1}$ and $Z_s = Z_{s,0}$. Note that $I_{s,i} = Z_{s,i} = 0$ if $s + i > r/2$. The symmetric group Sym_t acts on I_s from the right by permuting the first t nodes of the bottom row of a diagram. Thus I_s and Z_s are $(B_r(\delta), k\mathrm{Sym}_t)$-bimodules. Furthermore Z_s is a free right $k\mathrm{Sym}_t$-module which has as a basis the canonical images of the diagrams in which the vertical edges do not cross and of which the bottom row has precisely s horizontal edges which each join two consecutive nodes of the last $2s$ nodes. One easily checks that there are

$$\frac{r!}{s!t!2^s}$$

such diagrams.

Let λ be a partition of t and let $S(\lambda)$, $M(\lambda)$ and $Y(\lambda)$ be the Specht module, permutation module and Young module of $k\mathrm{Sym}_t$ associated to λ. If $\operatorname{char} k = 0$, then $S(\lambda)$ is irreducible and we also denote it by $D(\lambda)$. If $\operatorname{char} k = p > 0$ and λ is p-regular, then $S(\lambda)$ has a simple head and we denote it by $D(\lambda)$. Denote the sign representation of $k\mathrm{Sym}_t$ by k_{sg}.

Following [21] (see also [27]), we define the *Specht* (*or cell*) *module* $\mathcal{S}(\lambda)$ and *twisted Specht* (*or cell*) *module* $\widetilde{\mathcal{S}}(\lambda)$ for the Brauer algebra by

$$\mathcal{S}(\lambda) := Z_s \otimes_{k\mathrm{Sym}_t} S(\lambda) \quad \text{and}$$
$$\widetilde{\mathcal{S}}(\lambda) := Z_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes S(\lambda)\big).$$

By the above, $\dim \mathcal{S}(\lambda) = \dim \widetilde{\mathcal{S}}(\lambda) = \frac{r!}{s!t!2^s} \dim S(\lambda)$. By [25, Rem. 6.4] we have $k_{\mathrm{sg}} \otimes S(\lambda) \cong S(\lambda')^*$, where λ' denotes the transpose of λ. If $\operatorname{char} k = 0$ or $> t$, then $S(\lambda)^* \cong S(\lambda)$ and $\widetilde{\mathcal{S}}(\lambda) \cong \mathcal{S}(\lambda')$.

Remark 1 The definitions and results in [27] have obvious “twisted versions” and in what follows we will also cite [27] for those twisted versions.

Following Hartmann and Paget [27], we define the *permutation module* $\mathcal{M}(\lambda)$ and the *twisted permutation module* $\widetilde{\mathcal{M}}(\lambda)$ for the Brauer algebra by

$$\mathcal{M}(\lambda) := \mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} M(\lambda) \quad \text{and}$$
$$\widetilde{\mathcal{M}}(\lambda) := \mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} \big(k_{\mathrm{sg}} \otimes M(\lambda)\big).$$

Here $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r}$ is defined by $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V = I_s \otimes_{k\mathrm{Sym}_t} V$ for any $k\mathrm{Sym}_t$-module V. Note that $\widetilde{\mathcal{M}}((1^r)) \cong B_r$, since $k_{\mathrm{sg}} \otimes k\mathrm{Sym}_r \cong k\mathrm{Sym}_r$ as $k\mathrm{Sym}_r$-modules. If λ is p-regular and $\lambda \neq \emptyset$ in case r is even ≥ 2 and $\delta = 0$, then $\mathcal{S}(\lambda)$ and $\widetilde{\mathcal{S}}(\lambda)$ have a simple head which we denote by $\mathcal{D}(\lambda)$ and $\widetilde{\mathcal{D}}(\lambda)$. Whenever we write $\mathcal{D}(\lambda)$ and $\widetilde{\mathcal{D}}(\lambda)$ for some p-regular λ, we assume that $\lambda \neq \emptyset$ in case r is even ≥ 2 and $\delta = 0$.

In [10], after Lemma 2.1 and in Sect. 8, it is pointed out that it can be shown by completely elementary arguments that the above partitions form a labelling set for the irreducible B_r-modules. The exception $\lambda \neq \emptyset$ in case r is even ≥ 2 and $\delta = 0$ is caused by the fact that in case $\delta = 0$, B_2 has a one-dimensional nilpotent ideal with quotient isomorphic to $k\mathrm{Sym}_2$.

Finally, we define the *Young module* $\mathcal{Y}(\lambda)$ and the *twisted Young* module $\widetilde{\mathcal{Y}}(\lambda)$ for the Brauer algebra as the unique indecomposable summand of $\mathcal{M}(\lambda)$ (resp. $\widetilde{\mathcal{M}}(\lambda)$) which surjects onto $Z_s \otimes_{k\mathrm{Sym}_t} Y(\lambda)$ (resp. $Z_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes Y(\lambda)\big)$); see [27, Def. 6.3]. There it is also observed that $\mathcal{Y}(\lambda)$ and $\widetilde{\mathcal{Y}}(\lambda)$ are actually indecomposable summands of $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} Y(\lambda)$ and $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} \big(k_{\mathrm{sg}} \otimes Y(\lambda)\big)$.

Let i be an integer ≥ 0. The stabilizer of $\{\{1,2\},\dots,\{2i-1,2i\}\}$ in Sym_{2i} is isomorphic to the hyperoctahedral group of degree i and order $2^i i!$ and we denote it by H_i. We consider Sym_{2i} and H_i as embedded in Sym_t via the embedding $\mathrm{Sym}_{t-2i} \times \mathrm{Sym}_{2i} \subseteq \mathrm{Sym}_t$. From the proof of [27, Prop. 7.3] we deduce the following

Proposition 1.1 (cf. proof of [27, Prop. 7.3]) *Let V be a $k\mathrm{Sym}_t$-module.*

(i) $W := \mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V$ *has a descending filtration* $W = W_0 \supseteq W_1 \supseteq \cdots$ *such that* $W_i = 0$ *for* $i > \lfloor t/2 \rfloor$ *and* $W_i/W_{i+1} \cong Z_{s,i} \otimes_{k\mathrm{Sym}_t} V$ *for* $i \geq 0$.
(ii) $Z_{s,i} \otimes_{k\mathrm{Sym}_t} V \cong Z_{s+i} \otimes_{k\mathrm{Sym}_{t-2i}} V_{H_i}$ *for* $i \leq \lfloor t/2 \rfloor$, *where* V_{H_i} *is the largest trivial* H_i*-module quotient of* V.

The filtration of $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V = I_s \otimes_{k\mathrm{Sym}_t} V$ is constructed as follows. Let $I_s(i)$ be the subspace of I_s spanned by the diagrams of which the bottom row has s horizontal edges which each join two consecutive nodes of the last $2s$ nodes and has precisely i other horizontal edges. Then $I_{s,i} = \bigoplus_{j \geq i} I_s(j)$. Since each $I_s(i)$ is stable under the right action of Sym_t on I_s, we have $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V = \bigoplus_{i \geq 0} (I_s(i) \otimes_{k\mathrm{Sym}_t} V)$. Now we put $W_i = \bigoplus_{j \geq i} (I_s(j) \otimes_{k\mathrm{Sym}_t} V) \cong I_{s,i} \otimes_{k\mathrm{Sym}_t} V$ and observe that W_i is a B_r-submodule of W.

We record the following consequence of [10, Prop. 6.1] which was mentioned to us by A. Cox. It shows that we can restrict to the case that δ lies in the prime field. Of course, a sharper result is known in characteristic 0; see [6,40].

Proposition 1.2 (cf. [10, Prop. 6.1]) *Assume that δ does not lie in the prime field. Put $N_i = r!/\big(i!(r-2i)!2^i\big)$. Then*

$$B_r(\delta) \cong \bigoplus_{i=0}^{\lfloor r/2 \rfloor} \mathrm{Mat}_{N_i}(k\mathrm{Sym}_{r-2i}).$$

Proof Let $i \in \{0, \dots, \lfloor r/2 \rfloor\}$ and put $t_i = r - 2i$. Let J_i be the two-sided ideal of B_r that is spanned by the diagrams which have at least $2i$ horizontal edges. Note that $J_i = I_i B_r$. By [10, Prop. 6.1] we have that p-regular partitions of different numbers belong to different blocks. Since cell modules always belong to one block, we get that $\mathcal{S}(\lambda)$ can only have composition factors $\mathcal{D}(\mu)$, μ a p-regular partition of $|\lambda|$. From Proposition 1.1 we now deduce that $I_i = \mathrm{Ind}_{k\mathrm{Sym}_{t_i}}^{B_r} k\mathrm{Sym}_{t_i}$ has only composition factors $\mathcal{D}(\mu)$, μ a p-regular partition with $|\mu| \le t_i$. The same must hold for J_i, since it is a sum of images of I_i. By the proof of [27, Prop. 3.3] the irreducible module $\mathcal{D}(\mu)$, μ p-regular, is killed by I_i if and only if $|\mu| > t_i$. Since the composition factors of B_r/J_i are all killed by J_i they must be of the form $\mathcal{D}(\mu)$, μp-regular with $|\mu| > t_i$. By [10, Prop. 6.1] there exists a left ideal J^i of B_r such that $B_r = J_i \oplus J^i$. Since we are dealing with the left regular module it is clear that J^i must be a two-sided ideal. It follows that $B_r(\delta) \cong \bigoplus_{i=0}^{\lfloor r/2 \rfloor} J_i/J_{i+1}$, where each of the algebras J_i/J_{i+1} has a unit element. The algebra J_i/J_{i+1} is isomorphic to $\mathrm{Mat}_{N_i}(k\mathrm{Sym}_{t_i})$ where the multiplication is given by $A \circ B = AXB$ for some fixed $X \in \mathrm{Mat}_{N_i}(k\mathrm{Sym}_{t_i})$; see [5] and [32, Sect. 4]. Since J_i/J_{i+1} has a unit element we must have that X is invertible in $\mathrm{Mat}_{N_i}(k\mathrm{Sym}_{t_i})$. But then $A \mapsto AX$ defines an isomorphism $J_i/J_{i+1} \xrightarrow{\sim} \mathrm{Mat}_{N_i}(k\mathrm{Sym}_{t_i})$. □

In the remainder of this section we assume that $\delta = -n = -2m$ and that $m \ge r$. Note that if the field k is of prime characteristic $p > 2$, then any element of the prime subfield of k can be represented by an integer of this form. Under the representation $B_r \to \mathrm{End}_{\mathrm{Sp}_n}(E^{\otimes r})$ each Brauer diagram corresponds to an endomorphism of the Sp_n-module $E^{\otimes r}$. There is a more direct way to associate to each Brauer diagram an endomorphism of the Sp_n-module $E^{\otimes r}$; see, e.g. [4, p. 871] or [39, Sect. 3]. Furthermore, there is a unique algebra structure on the vector space B_r such that this other map is a homomorphism of algebras. Let us call the resulting algebra the symplectic Brauer algebra and denote it by $\widetilde{B}_r(n)$ or just $\widetilde{B}_r$. Of course, the other map comes from an isomorphism $\widetilde{B}_r(n) \xrightarrow{\sim} B_r(-n)$. This isomorphism sends each of the r standard generators of $\widetilde{B}_r$ to the negative of the corresponding standard generator of B_r. This implies that each diagram $d \in \widetilde{B}_r(n)$ corresponds to $\pm d \in B_r(-n)$. The multiplication of $\widetilde{B}_r$ is more complicated to describe. In [4] Brauer introduced the algebras $B_r(n)$ and $\widetilde{B}_r(n)$ and their action on tensor space separately, the isomorphism $\widetilde{B}_r(n) \xrightarrow{\sim} B_r(-n)$ was observed later in [26]. In [39, Sect. 3] it is explained, using Brauer's original arguments, that, more generally, $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t_2}, E^{\otimes t_1})$ has a basis indexed by (t_1, t_2)-diagrams. These are diagrams which are graphs whose vertices are arranged in two rows, t_1 in the top row and t_2 in the bottom row, and whose edges form a matching of the $t_1 + t_2$ nodes in (unordered) pairs. The horizontal edges in the bottom row correspond to contractions by means of the symplectic form and the horizontal edges in the top row correspond to "multiplications" by the symplectic invariant $\sum_{i=1}^{n} \epsilon_i e_i \otimes e_{i'}$. In the proofs of Lemmas 2.3 and 2.4 below we will use these diagram bases.

The symplectic form on E induces a nondegenerate bilinear form on $E^{\otimes r}$. So $\mathrm{End}_k(E^{\otimes r})$ has a transpose map. Recall that B_r has a standard anti-automorphism ι that flips a diagram over the horizontal axis. One easily checks that $\iota(b)$ acts as the transpose of b for all $b \in B_r$. This means that the B_r-module $E^{\otimes r}$ is self-dual. Under the isomorphism $B_r \xrightarrow{\sim} \widetilde{B}_r$ the left ideal I_s is mapped to a left ideal $\widetilde{I}_s$ of $\widetilde{B}_r$ which is spanned by the same diagrams. These diagrams are in 1–1 correspondence with the (r, t)-diagrams: just omit the last $2s$ nodes in the bottom row and the edges which have these nodes as endpoints. So the canonical isomorphism $\widetilde{B}_r \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r})$ induces a canonical isomorphism

$$I_s \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, E^{\otimes r}) \otimes k_{\mathrm{sg}}$$

of $(B_r, k\mathrm{Sym}_t)$-bimodules. The vector space $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})$ has a natural $(k\mathrm{Sym}_t, B_r)$-bimodule structure and therefore, by means of the standard anti-automorphisms of Sym_t and B_r, also a natural $(B_r, k\mathrm{Sym}_t)$-bimodule structure. Composing the above isomorphism with the transpose map $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, E^{\otimes r}) \to \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})$ we obtain an canonical isomorphism

$$\varphi : I_s \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes k_{\mathrm{sg}} \tag{1}$$

of $(B_r, k\mathrm{Sym}_t)$-bimodules, which induces an isomorphism

$$Z_s \xrightarrow{\sim} \big(\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})/\varphi(I_{s,1})\big) \otimes k_{\mathrm{sg}} \tag{2}$$

of $(B_r, k\mathrm{Sym}_t)$-bimodules.

2 The symplectic Schur functor

For a finite dimensional algebra A over k, we denote the category of finite dimensional A-modules by $\mathrm{mod}(A)$. The category $\mathrm{mod}(S(n, r))$ can be identified with the category of GL_n-modules whose coefficients are homogeneous polynomials of degree r in the matrix entries. Assume that $n \geq r \geq 0$. The Schur functor $f : \mathrm{mod}(S(n, r)) \to \mathrm{mod}(k\mathrm{Sym}_r)$ can be defined by

$$f(M) = \mathrm{Hom}_{S(n,r)}(E^{\otimes r}, M) = \mathrm{Hom}_{\mathrm{GL}_n}(E^{\otimes r}, M).$$

Here the action of the symmetric group comes from the action on $E^{\otimes r}$ and we use the inversion to turn right modules into left modules. An equivalent definition is: $f(M) = M_{\varpi_r}$, the weight space corresponding to the weight $\varpi_r = (1^r) = (1, 1, \dots, 1) \in \mathbb{Z}^r \subseteq \mathbb{Z}^n$; see [25]. The isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_n}(E^{\otimes r}, M) \xrightarrow{\sim} M_{\varpi_r} \tag{3}$$

is given by $u \mapsto u(e_1 \otimes e_2 \otimes \cdots \otimes e_r)$. This can be deduced from [25, 6.2g Rem. 1 and 6.4f]. We have embeddings $\mathrm{Sym}_r \subseteq \mathrm{Sym}_n \subseteq N_{\mathrm{GL}_n}(T)$, where the second embedding is by permutation matrices. Then ϖ_r is fixed by Sym_r, so there is an action of Sym_r on M_{ϖ_r} for every $S(n, r)$-module M. With this action (3) is Sym_r-equivariant. The inverse Schur functor $g : \mathrm{mod}(k\mathrm{Sym}_r) \to \mathrm{mod}(S(n, r))$ can be defined by

$$g(V) = E^{\otimes r} \otimes_{k\mathrm{Sym}_r} V.$$

We now retain the notation and assumptions of Sect. 1.2. So $n = 2m$, $m \geq r$ and $B_r = B_r(-n)$. We define the *symplectic Schur functor*

$$f_0 : \mathrm{mod}(S_0(n, r)) \to \mathrm{mod}(B_r)$$

by

$$f_0(M) = \mathrm{Hom}_{S_0(n,r)}(E^{\otimes r}, M) = \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, M).$$

Here the action of the Brauer algebra comes from the action on $E^{\otimes r}$ and we use the standard anti-automorphism of B_r to turn right modules into left modules. Note that the action of Sym_r on $E^{\otimes r}$ inherited from that of B_r is its natural action twisted by the sign. Since $E = \nabla_0(\varepsilon_1) = \Delta_0(\varepsilon_1)$ is a tilting module, the same holds for $E^{\otimes r}$; see, e.g. [18, Prop. 1.2]. This implies that f_0 maps exact sequences of modules with a good filtration to exact sequences.

We define the *inverse symplectic Schur functor*

$$g_0 : \mathrm{mod}(B_r) \to \mathrm{mod}(S_0(n, r))$$

by

$$g_0(V) = E^{\otimes r} \otimes_{B_r} V.$$

By [36, Thm 2.11] we have for $V \in \mathrm{mod}(B_r)$ and $M \in \mathrm{mod}(S_0(n, r))$

$$\mathrm{Hom}_{\mathrm{Sp}_n}(g_0(V), M) \cong \mathrm{Hom}_{B_r}(V, f_0(M)). \tag{4}$$

There is an alternative for f_0 and g_0:

$$\tilde{f}_0(M) = E^{\otimes r} \otimes_{S_0(n,r)} M \quad \text{and} \quad \tilde{g}_0(V) = \mathrm{Hom}_{B_r}(E^{\otimes r}, M).$$

But, by [36, Lemma 3.60], we have $\tilde{f}_0(V^*) \cong f_0(V)^*$ and $\tilde{g}_0(V^*) \cong g_0(V)^*$. So the results obtained using $\tilde{f}_0$ and $\tilde{g}_0$ can also be obtained by dualizing the results obtained using f_0 and g_0. We sketch a proof of the following exercise in Brauer's Formula.

Lemma 2.1 *Let λ be a partition of $t = r - 2s$. Then*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_n}(\Delta_0(\lambda), E^{\otimes r}) = \dim \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, \nabla_0(\lambda)) = \frac{r!}{s!t!2^s} \dim S(\lambda).$$

Proof First note that, since $E^{\otimes r}$ has a good filtration, as an Sp_n-module, the dimension of $\mathrm{Hom}_{\mathrm{Sp}_n}(\Delta_0(\lambda), E^{\otimes})$ is equal to the multiplicity of $\nabla_0(\lambda)$ in a good filtration of $E^{\otimes r}$. Similar remarks apply to the dimension of $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, \nabla_0(\lambda))$. For a partition λ, with at most m parts, we write $\chi_0(\lambda)$ for the formal character of $\nabla_0(\lambda)$. We have to show that the coefficient of $\chi_0(\lambda)$ in an expression of $\mathrm{ch}\, E^{\otimes r}$ as a $\mathbb{Z}$-linear combination of Weyl characters, is $\frac{r!}{s!t!2^s} \dim S(\lambda)$. For a partition λ of r we denote by χ^λ the corresponding character of Sym_r. Then, for $r \geq 1$, by the branching rule, we have

$$\chi^\lambda \downarrow^{\mathrm{Sym}_{r+1}}_{\mathrm{Sym}_r} = \sum_\mu \chi^\mu,$$

where the sum is over all partitions μ of r such that the diagram of μ is obtained from that of λ by the removal of one box. By Frobenius reciprocity we have

$$\chi^\lambda \uparrow^{\mathrm{Sym}_r}_{\mathrm{Sym}_{r-1}} = \sum_\mu \chi^\mu,$$

where μ ranges over those partitions of r whose diagram is obtained by adding one box. In particular the degree of χ^λ is given by

$$\begin{aligned} r \deg(\chi^\lambda) &= \deg(\chi^\lambda \uparrow^{\mathrm{Sym}_r}_{\mathrm{Sym}_{r-1}}) \\ &= \sum_\mu \deg(\chi^\mu). \end{aligned}$$

Now we define $\psi_0 = 1$ and for $1 \leq r \leq m$ we define

$$\psi_r = \sum_\lambda \deg(\chi^\lambda)\chi_0(\lambda),$$

where the sum is over all partitions of r. We claim that, for $1 \leq r < m$, we have

$$\chi_0(1)\psi_r = \psi_{r+1} + r\psi_{r-1}. \tag{*}$$

By Brauer's Formula (see, e.g. [29, Lem. II.5.8 b)]) we have

$$\chi_0(1)\psi_r = \sum_{i=1}^{m}\sum_{\mu}\deg(\chi^{\mu})\chi_0(\mu+\epsilon_i) + \sum_{i=1}^{m}\sum_{\mu}\deg(\chi^{\mu})\chi_0(\mu-\epsilon_i),$$

where in both sums μ ranges over partitions of r. One easily checks that $\chi_0(\mu\pm\epsilon_i)\neq 0$ if and only if $\mu\pm\epsilon_i$ is a partition. For a partition λ of $r+1$, we see that the coefficient of $\chi_0(\lambda)$ in $\chi_0(1)\psi_r$ is $\sum_{\mu}\deg(\chi^{\mu})$, where μ ranges over partitions of r such that the diagram of μ is obtained from the diagram of λ by removing a box. So this coefficient is $\deg(\chi^{\lambda})$. For a partition λ of $r-1$, the coefficient of $\chi_0(\lambda)$ in $\chi_0(1)\psi_r$ is $\sum_{\mu}\deg(\chi^{\mu})$, where μ ranges over all partitions of r such that the diagram of μ is obtained from the diagram of λ by adding a box. So the coefficient of $\chi_0(\lambda)$ is $r\deg(\chi^{\lambda})$. This proves (*).

We leave it to the reader to use (*) to prove by induction that

$$\begin{aligned}\chi_0(1)^r &= \psi_r + \frac{r(r-1)}{2}\psi_{r-2} + \cdots\\ &= \sum_{s=0}^{\lfloor r/2\rfloor}\frac{r!}{2^s s!(r-2s)!}\psi_{r-2s}.\end{aligned}$$

From this we get as required that the coefficient $a_r(\lambda)$ in the expression

$$\chi_0(1)^r = \sum_{\lambda} a_r(\lambda)\chi_0(\lambda)$$

is $\dfrac{r!}{2^s s!t!}\deg(\chi^{\lambda})$, where $|\lambda| = t = r-2s$. □

Recall that induced modules for a reductive group can be realized in the algebra of regular functions of the group. Let λ be a partition of r. In [16, Prop. 1.4] it was proved that restriction of functions induces an epimorphism $\nabla(\lambda)\to\nabla_0(\lambda)$ of Sp_n-modules. Now we can form a commutative diagram as below where the vertical maps are induced by the restriction of functions $\nabla(\lambda)\to\nabla_0(\lambda)$ and the horizontal maps are evaluation at $e_1\otimes\cdots\otimes e_r$.

$$\begin{array}{ccc}\mathrm{Hom}_{\mathrm{GL}_n}(E^{\otimes r},\nabla(\lambda)) & \longrightarrow & \nabla(\lambda)_{\varpi_r}\\ \downarrow & & \downarrow\\ \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r},\nabla_0(\lambda)) & \longrightarrow & \nabla_0(\lambda)_{\varpi_r}\end{array}\quad . \tag{5}$$

Here $\nabla(\lambda)_{\varpi_r}$ denotes the ϖ_r-weight space of $\nabla(\lambda)$ with respect to T and $\nabla_0(\lambda)_{\varpi_r}$ denotes the ϖ_r-weight space of $\nabla_0(\lambda)$ with respect to T_0.

Lemma 2.2

(i) *Let M be a homogeneous polynomial T-module of degree r and let $\mu\in\mathbb{N}^m$ with $|\mu| = r$. Then the μ-weight space of M with respect to T is the same as that with respect to T_0.*
(ii) *Let λ be a partition of r and let $\mu\in\mathbb{N}^m$ with $|\mu| = r$. The restriction of functions $\nabla(\lambda)_\mu\to\nabla_0(\lambda)_\mu$ on the μ-weight spaces with respect to T_0 is an isomorphism.*
(iii) *All maps in (5) are isomorphisms.*

Proof (i) A weight μ of T vanishes on T_0 if and only if $\mu_i = \mu_{i'}$ for all $i \in \{1, \dots, n\}$. So if μ and ν are weights of T such that μ is polynomial, $\nu \in \mathbb{Z}^m$, $|\mu| = |\nu|$ and $\mu|_{T_0} = \nu|_{T_0}$, then $\mu = \nu$.
(ii) Clearly $\nabla(\lambda) \to \nabla_0(\lambda)$ induces a surjection on the weight spaces for T_0. So it suffices to show that $\nabla(\lambda)_\mu$ and $\nabla_0(\lambda)_\mu$ have the same dimension. Note that, by (i), $\nabla(\lambda)_\mu$ is also the μ-weight space with respect to T. Let $\mu \in \mathbb{N}^m$ with $|\mu| = r$. By [25, 4.5a] $\dim \nabla(\lambda)_\mu$ is the number of standard λ-tableaux of content μ and by [30, Sect. 4] (or [16, Thm. 2.3b]) $\dim \nabla_0(\lambda)_\mu$ is the number of symplectic standard λ-tableaux of which the content ν satisfies $\overline{\nu} = \mu$. Here a tableau is called *symplectic standard* if it is standard for the ordering $1' < 1 < 2' < 2 \cdots < m' < m$ of $\{1, \dots, n\}$ and if for each $i \in \{1, \dots, m\}$, i and i' only occur in the first i rows. The second condition is vacuous if the content ν is in $\mathbb{N}^m$, since then $m+1, \dots, n$ don't occur in a λ-tableau of content ν. Since $\mu \in \mathbb{N}^m$, we have that $\overline{\nu} = \mu$ implies $\nu = \mu$ by the proof of (i). So the two dimensions are the same.
(iii) That the horizontal map in the top row of (5) is an isomorphism was pointed out before; see (3). The vertical map on the right is an isomorphism by (ii). It follows that the horizontal map in the bottom row is surjective. But then it must be an isomorphism by Lemma 2.1. Now the vertical map on the left must also be an isomorphism, since it is a composite of isomorphisms. □

For $\lambda \in \mathbb{N}^l$ we put $S^\lambda E = S^{\lambda_1} E \otimes \cdots \otimes S^{\lambda_l} E$ and $\bigwedge^\lambda E = \bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_l} E$.

Lemma 2.3 *Recall that $t = r - 2s$. The following holds.*

(i) *Let λ be a partition of t. Then the canonical homomorphism*

$$\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes_{k\mathrm{Sym}_t} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda)) \to \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, \nabla_0(\lambda)),$$

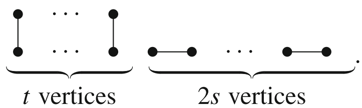
given by composition, is surjective.

(ii) *Let M be an $S(n,t)$-module. The canonical homomorphism*

$$\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes_{k\mathrm{Sym}_t} \mathrm{Hom}_{\mathrm{GL}_n}(E^{\otimes t}, M) \to \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, M),$$

given by composition, is an isomorphism if M is a direct sum of direct summands of $E^{\otimes t}$ and it is surjective if M is injective.

Proof (i) By Lemma 2.1 it suffices to give a family of $\frac{r!}{s!t!2^s} \dim S(\lambda)$ elements of $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes_{k\mathrm{Sym}_t} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda))$ which is mapped to an independent family in $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, \nabla_0(\lambda))$. As pointed out before, $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})$ has a basis indexed by (t, r)-diagrams. Let D be the set of (t, r)-diagrams that have no horizontal edges in the top row and whose vertical edges do not cross, and let $(p_d)_{d \in D}$ be the corresponding family of basis elements in $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})$. Let $(u_i)_{\in I}$ be a basis of $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda))$. We have $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda)) \cong S(\lambda)$ by Lemma 2.2(iii) (with $r = t$), $|D| = \frac{r!}{s!t!2^s}$ and $p_d \otimes u_i$ is mapped to $u_i \circ p_d$. So it suffices to show that the elements $u_i \circ p_d$, $d \in D$, $i \in I$, are linearly independent. So assume $\sum_{i,d} a_{id}\, u_i \circ p_d = 0$ for certain $a_{id} \in k$. Consider the following diagram $d_0 \in D$:

$d_0 =$ 

Put

$$v_0 = e_1 \otimes \cdots \otimes e_t \otimes e_{t+1} \otimes e_{(t+1)'} \otimes \cdots \otimes e_{t+s} \otimes e_{(t+s)'}.$$

Then we have for $d \in D$ that $p_d(v_0) = e_1 \otimes \cdots \otimes e_t$ if $d = d_0$ and 0 otherwise. It follows that $\sum_i a_{id_0} u_i(e_1 \otimes \cdots \otimes e_t) = 0$. By Lemma 2.2(iii) evaluation at $e_1 \otimes \cdots \otimes e_t$ is injective on $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda))$, so $a_{id_0} = 0$ for all $i \in I$. Since we can construct a similar vector for any other $d \in D$ it follows that $a_{id} = 0$ for all $i \in I$ and $d \in D$.
(ii) The class of $S(n,t)$-modules M for which this homomorphism is an isomorphism, is closed under taking direct summands and direct sums. The same holds for the class of $S(n,t)$-modules M for which this homomorphism is surjective. By [18, Lem. 3.4(i)] every injective $S(n,t)$-module is a direct sum of direct summands of some $S^\lambda E$, $\lambda \in \Lambda^+(n,t)$. Furthermore, $\mathrm{End}_{\mathrm{GL}_n}(E^{\otimes t}) \cong k\mathrm{Sym}_t$. So it suffices now to show that the homomorphism is surjective if $M = S^\lambda E$, $\lambda \in \Lambda^+(n,t)$.

Denote $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t})$ by $\mathcal{H}$ and the Schur functor $\mathrm{Hom}_{\mathrm{GL}_n}(E^{\otimes t}, -)$ by f. Let $0 \to M \to N \to P \to 0$ be a short exact sequence of $S(n,t)$-modules with a good filtration. Then we have the following diagram

$$\begin{array}{ccccccc}
\mathcal{H} \otimes_{k\mathrm{Sym}_t} f(M) & \longrightarrow & \mathcal{H} \otimes_{k\mathrm{Sym}_t} f(N) & \longrightarrow & \mathcal{H} \otimes_{k\mathrm{Sym}_t} f(P) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
f_0(M) & \longrightarrow & f_0(N) & \longrightarrow & f_0(P) & \longrightarrow & 0
\end{array}$$

with rows exact, because f is exact and f_0 is exact on modules with a good filtration. Here we have used that a GL_n-module with a good GL_n-filtration, also has a good Sp_n-filtration; see [17, App. A]. We deduce that if the homomorphism in (ii) is surjective for N, then it is surjective for P. Since the kernel of the canonical epimorphism $E^{\otimes t} \to S^\lambda E$ has a good GL_n-filtration by [19, 2.1.15(ii)(b)], we are done. □

In the theorem below f denotes the Schur functor from $\mathrm{mod}(S(n,t))$ to $\mathrm{mod}(k\mathrm{Sym}_t)$. Note that the second isomorphism in assertion (i) implies that when $\mathrm{char}\, k \neq 2$ and M is an injective $S(n,t)$-module, the homomorphism in Lemma 2.3(ii) is an isomorphism.

Theorem 2.1 *Recall that $m \geq r$. The following holds.*

(i) *For $\lambda \in \Lambda_0^+(m,r)$ we have*

$$\begin{aligned}
f_0(\nabla_0(\lambda)) &\cong \widetilde{\mathcal{S}}(\lambda), \\
f_0(S^\lambda E) &\cong \widetilde{\mathcal{M}}(\lambda) \quad \text{if } \mathrm{char}\, k \neq 2, \quad \text{and} \\
f_0\Big(\textstyle\bigwedge^\lambda E\Big) &\cong \mathcal{M}(\lambda) \quad \text{if } \mathrm{char}\, k = 0 \quad \text{or} > |\lambda|.
\end{aligned}$$

(ii) *Let M be an $S(n,t)$-module. If M is a direct sum of direct summands of $E^{\otimes t}$ or if $\mathrm{char}\, k \neq 2$ and M is injective, then*

$$f_0(M) \cong \mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r}\big(k_{\mathrm{sg}} \otimes f(M)\big).$$

Proof If we give $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes t}, \nabla_0(\lambda))$ the $k\mathrm{Sym}_t$-module structure coming from the action of Sym_t on $E^{\otimes t}$ by place permutations, then the isomorphisms in (5) are Sym_t-equivariant. Now Lemma 2.3(i) and the isomorphism (1) give us an epimorphism $I_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes S(\lambda)\big) \to f_0(\nabla_0(\lambda))$, since $\big(I_s \otimes k_{\mathrm{sg}}\big) \otimes_{k\mathrm{Sym}_t} S(\lambda) \cong I_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes S(\lambda)\big)$. The image of a nonzero homomorphism from $E^{\otimes r}$ to $\nabla_0(\lambda)$ must contain $L_0(\lambda)$ and therefore have λ as a weight. The image of a homomorphism in $\varphi(I_{s,1})$ does not have λ as a weight, since $\varphi(I_{s,1})$ has a basis of homomorphisms whose image lies is a submodule of $E^{\otimes t}$ which is isomorphic to $E^{\otimes(t-2)}$.

So, by (2) and the definition of $\widetilde{S}(\lambda)$, we obtain an epimorphism $\widetilde{S}(\lambda) \to f_0(\nabla_0(\lambda))$. By Lemma 2.1 this must be an isomorphism.

Let M be an $S(n,t)$-module. Lemma 2.3(ii) and the isomorphism φ give us a homomorphism

$$\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r}\big(k_{\mathrm{sg}} \otimes f(M)\big) \to f_0(M), \tag{*}$$

which is an isomorphism if M is a direct sum of direct summands of $E^{\otimes t}$ and surjective for M injective. Let $\lambda \in \Lambda_0(n,r)$ be a partition of t. Then we obtain an epimorphism $\widetilde{\mathcal{M}}(\lambda) \to f_0(S^\lambda E)$ and a homomorphism $\mathcal{M}(\lambda) \to f_0(\bigwedge^\lambda E)$, since $f(S^\lambda E) = M(\lambda)$ and $f(\bigwedge^\lambda E) = k_{\mathrm{sg}} \otimes M(\lambda)$ by [18, Lemma 3.5]. If char $k = 0$ or $> t$, then $S(n,t)$ is semisimple, so every $S(n,t)$-module is a direct sum of direct summands of $E^{\otimes t}$ and (*) is an isomorphism for every $S(n,t)$-module M. In particular, we have the third isomorphism in (i).

Now assume that char $k \neq 2$. We want to show that the epimorphism $\widetilde{\mathcal{M}} \to f_0(S^\lambda E)$ is an isomorphism. Since (*) is an isomorphism if char $k = 0$ it suffices to show that the dimensions of $f_0(S^\lambda E)$ and $\widetilde{\mathcal{M}}$ are independent of the characteristic. The dimension of $f_0(S^\lambda E)$ is independent of the characteristic, since, by [19, Prop. A.2.2(ii)], it only depends on the formal characters of the Sp_n-modules $E^{\otimes r}$ and $S^\lambda E$ (and these are independent of the characteristic). That $\widetilde{\mathcal{M}}$ has dimension independent of the characteristic follows from Proposition 1.1, the fact that $M(\lambda)$ is self dual and the following fact, the proof of which we leave to the reader.

Assume char $k \neq 2$. *Let G be a finite group with a* (*possibly trivial*) *sign homomorphism* $\mathrm{sg} : G \to \{\pm 1\}$, *let V be a permutation module for G over k with G-stable basis S. Then the dimension of* $(k_{\mathrm{sg}} \otimes V)^G$ *is equal to the number of G-orbits in S for which one* (*and therefore each*) *stabilizer is contained in* $\mathrm{Ker}(\mathrm{sg})$.

We have now proved the second isomorphism in (i) and we have also proved (ii), since every injective $S(n,t)$-module is a direct sum of direct summands of some $S^\lambda E$, $\lambda \in \Lambda^+(n,t)$.

□

Note that it follows from Theorem 2.1(i) that f_0 maps good filtrations to twisted Specht filtrations.

For $\lambda \in \Lambda_0^+(m,r)$ we denote the indecomposable tilting module for Sp_n of highest weight λ by $T_0(\lambda)$ and for λ p-regular we denote the projective cover of the irreducible B_r-module $\widetilde{\mathcal{D}}(\lambda)$ by $\widetilde{\mathcal{P}}(\lambda)$.

Proposition 2.1 *Let $\lambda \in \Lambda_0^+(m,r)$. Then $T_0(\lambda)$ is a direct summand of the* Sp_n*-module $E^{\otimes r}$ if and only if λ is p-regular and $\lambda \neq \emptyset$ in case r is even ≥ 2 and $\delta = 0$. Now assume that λ satisfies these conditions. Then*

(i) $f_0(T_0(\lambda)) = \widetilde{\mathcal{P}}(\lambda)$.
(ii) *The multiplicity of $T_0(\lambda)$ in $E^{\otimes r}$ is* $\dim \widetilde{\mathcal{D}}(\lambda)$.
(iii) *The decomposition number $[\widetilde{\mathcal{S}}(\mu) : \widetilde{\mathcal{D}}(\lambda)]$ is equal to the Δ-filtration multiplicity $(T_0(\lambda) : \Delta_0(\mu))$ and to the ∇-filtration multiplicity $(T_0(\lambda) : \nabla_0(\mu))$.*

Proof Let Ω be the set of all partitions satisfying the stated conditions. The symplectic Schur functor f_0 induces a category equivalence between the direct sums of direct summands of the Sp_n-module $E^{\otimes r}$ and the projective B_r-modules; see, e.g. [2, Prop 2.1(c)]. Clearly, the number of isomorphism classes of indecomposable B_r-projectives is equal to $|\Omega|$. So, to prove the first assertion, it suffices to show that for each $\lambda \in \Omega$, $T_0(\lambda)$ is a direct summand of $E^{\otimes r}$. By Lemma 1.1 we may assume that $t = r$. The indecomposable GL_n tilting module $T(\lambda)$ is a direct summand of $E^{\otimes r}$, for example by [19, Sect. 4.3, (1) and (4)]. Moreover, as an Sp_n-module, $T(\lambda)$ is also a tilting module and has unique highest weight λ, and λ occurs with multiplicity 1. Thus we have $T(\lambda) \cong T_0(\lambda) \oplus Y$, where Y is a direct sum of indecomposable tilting modules for Sp_n, of weight less than λ. In particular, $T_0(\lambda)$ occurs as a component of $E^{\otimes r}$.

Now let $\lambda \in \Omega$. By Theorem 2.1(i) we have that $f_0(T_0(\lambda))$ surjects onto $f_0(\nabla_0(\lambda)) = \widetilde{\mathcal{S}}(\lambda)$. But $\widetilde{\mathcal{S}}(\lambda)$ surjects onto $\widetilde{\mathcal{D}}(\lambda)$. This proves (i), and (ii) is now also clear, since this multiplicity (as an indecomposable direct summand) is equal to the multiplicity of $\widetilde{\mathcal{P}}(\lambda)$ in B_r. We have $g_0(f_0(M)) \cong M$ canonically for $M = E^{\otimes r}$ and therefore also for $M = T_0(\lambda)$. By (4) we have $\mathrm{Hom}_{\mathrm{Sp}_n}(T_0(\lambda), M) \cong \mathrm{Hom}_{B_r}(\widetilde{\mathcal{P}}(\lambda), f_0(M))$ for every $S_0(n,r)$-module M. So $[\widetilde{\mathcal{S}}(\mu) : \widetilde{\mathcal{D}}(\lambda)] = \dim \mathrm{Hom}_{B_r}(\widetilde{\mathcal{P}}(\lambda), \widetilde{\mathcal{S}}(\mu)) = \dim \mathrm{Hom}_{\mathrm{Sp}_n}(T_0(\lambda), \nabla_0(\mu)) = (T_0(\lambda) : \Delta_0(\mu))$. The equality $(T_0(\lambda) : \Delta_0(\mu)) = (T_0(\lambda) : \nabla_0(\mu))$ follows from the fact that both multiplicities are equal to the coefficient of the Weyl character $\chi_0(\mu)$ in $\mathrm{ch}\, T_0(\lambda)$. □

We now combine Proposition 2.1 with a result of Adamovich and Rybnikov. For this we need the following notation. Let m' be a positive integer and let λ be a partition with $l(\lambda) \le m$ and $l(\lambda') = \lambda_1 \le m'$, that is, a partition of which the diagram fits into an $m \times m'$-rectangle. Here λ' denotes the transpose of λ. Then we define

$$\lambda^\dagger = (m - \lambda'_{m'}, m - \lambda'_{m'-1}, \dots, m - \lambda'_1).$$

So $\lambda^\dagger$ is the transpose of the complement of λ in the $m \times m'$-rectangle. In particular, $l(\lambda^\dagger) \le m'$ and $\lambda^\dagger_1 \le m$.

Corollary *Let $\lambda, \mu \in \Lambda_0^+(r,r)$ with λ p-regular. Assume that $\lambda_1, \mu_1 \le m'$. Then we have the equality of decomposition numbers*

$$[\widetilde{\mathcal{S}}(\mu) : \widetilde{\mathcal{D}}(\lambda)] = [\nabla'_0(\lambda^\dagger) : L'_0(\mu^\dagger)],$$

where ∇'_0 and L'_0 denote induced and irreducible modules for $\mathrm{Sp}_{2m'}$.

Proof This follows immediately from Proposition 2.1(iii) and [1, Cor 2.4]. □

Remarks 2.1 1. Let f_0^t denote the symplectic Schur functor from $\mathrm{mod}(S_0(n,t))$ to $\mathrm{mod}(B_t)$ and let M be an Sp_n-module which has a filtration with sections isomorphic to some $\nabla_0(\lambda)$, λ a partition of t. Then

$$f_0(M) \cong Z_s \otimes_{k\mathrm{Sym}_t} f_0^t(M).$$

This is shown as follows. First we construct a homomorphism

$$\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes_{k\mathrm{Sym}_t} f_0^t(M) \to f_0(M)$$

by means of function composition. Then we form, for a short exact sequence $0 \to M \to N \to P \to 0$ of Sp_n-modules of the above type, a diagram as in the proof of Lemma 2.3(ii) with f replaced by f_0^t and deduce that if the homomorphism is surjective for M and P, then also for N. We then obtain surjectivity by induction on the length of a good filtration. Then we factor out $\varphi(I_{s,1})$ as in the proof of the first isomorphism of Theorem 2.1(i) and finish by showing that the dimensions are equal.

2. Let M be an $S(n,r)$-module. Put $\pi_r = \{\lambda \in \Lambda_0^+(m,r) \mid |\lambda| < r\}$ and let $N = O_{\pi_r,0}(M)$ be the largest Sp_n-submodule of M which belongs to π_r, i.e. which has only composition factors $L(\lambda)$, $\lambda \in \pi_r$. By [19, Prop. A2.2(v), Lem. A3.1] N has a filtration with sections

$\nabla_0(\lambda)$, $\lambda \in \pi_r$, and M/N has a filtration with sections $\nabla_0(\lambda)$, λ a partition of r. Note that $M/N = \nabla_0(\lambda)$ if $M = \nabla(\lambda)$. Now we can form the diagram

$$\begin{array}{ccc} f(M) & \longrightarrow & M_{\varpi_r} \\ \downarrow & & \downarrow \\ f_0(M/N) & \longrightarrow & (M/N)_{\varpi_r} \end{array}$$

in the same way as (5) and by a proof very similar to that of Lemma 2.2(iii) we show that all maps are isomorphisms. For the isomorphism $f(M) \xrightarrow{\sim} f_0(M/N)$ to be B_r-equivariant one needs to twist $f(M)$ with the sign.

3. It is easy to see that the canonical homomorphism $\mathcal{M}(\lambda) \to f_0(\bigwedge^\lambda E)$ constructed in the proof of Theorem 2.1 is not always an isomorphism. Assume that $r \geq 2p$ and take $\lambda = (r)$. Then $\mathcal{M}(\lambda) = B_r \otimes_{k\mathrm{Sym}_r} k$ is a cyclic B_r-module with generator $1 \otimes 1$ which is not killed by any diagram in B_r. But its image in $f_0(\bigwedge^\lambda E)$ is the canonical projection $P : E^{\otimes r} \to \bigwedge^r E$ which is killed by any diagram with at least p horizontal edges in a row, since the pth power of the symplectic invariant is zero in the exterior algebra $\bigwedge E$.

4. We have the adjoint isomorphism

$$\mathrm{Hom}_{S_0(n,r)}(V \otimes_{k\mathrm{Sym}_t} E^{\otimes t}, E^{\otimes r}) \cong \mathrm{Hom}_{k\mathrm{Sym}_t}(V, \mathrm{Hom}_{S_0(n,r)}(E^{\otimes t}, E^{\otimes r}))$$

for every Sym_t-module V; see, e.g. [36, Thm. 2.11]. From this we deduce that for every $S(n,t)$ module M with $g(f(M)) \cong M$ canonically, we have $f_0(M^*) \cong \mathrm{Hom}_{k\mathrm{Sym}_t}(f(M), k_{\mathrm{sg}} \otimes I_s)$ and $f_0(M) \cong \mathrm{Hom}_{k\mathrm{Sym}_t}(k_{\mathrm{sg}} \otimes I_s^*, f(M))$. In particular we obtain for a partition λ of t, $f_0(\bigwedge^\lambda E) \cong \mathrm{Hom}_{k\mathrm{Sym}_t}(M(\lambda), I_s)$. Unfortunately, we have been unable to make effective use of this isomorphism.

Lemma 2.4 *Let M be an Sp_n-module. Then the canonical homomorphism*

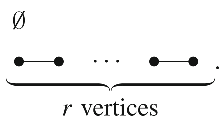
$$E^{\otimes r} \otimes_{B_r} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, M) \to M$$

given by function application is an isomorphism if M is a direct summand of $E^{\otimes r}$ or if r is even ≥ 4 and $M = k$.

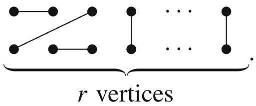
Proof That the canonical homomorphism is an isomorphism under the first condition is obvious, since $\mathrm{End}_{\mathrm{Sp}_n}(E^{\otimes r}) \cong B_r$. So assume r is even ≥ 4 and $M = k$. Since the homomorphism is always surjective and

$$E^{\otimes r} \otimes_{B_r} \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k) \cong \mathrm{Hom}_{B_r}(\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k), E^{\otimes r})^*$$

by [36, Lemma 3.60], it suffices to show that $\mathrm{Hom}_{B_r}(\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k), E^{\otimes r})$ is one-dimensional. Recall that $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k)$ is a left B_r-module by means of the standard anti-automorphism ι of B_r. It has a basis indexed by $(0,r)$-diagrams and it is generated as a $k\mathrm{Sym}_r$-module by the homomorphism P corresponding to the $(0,r)$-diagram



It follows that any B_r-homomorphism from $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k)$ to $E^{\otimes r}$ is determined by its image of P. One easily checks that $P \circ \iota(d) = \pm P$, where $d \in B_r$ is given by

$d =$ 

Therefore the image of P must lie in $d \cdot E^{\otimes r} = ku \otimes E^{\otimes(r-2)}$, where u is the invariant $\sum_{i=1}^{n} \epsilon_i e_i \otimes e_{i'}$. Similarly we find that it must lie in $E^{\otimes i} \otimes ku \otimes E^{\otimes(r-i-2)}$ for any even integer i with $0 \leq i \leq r-2$. We conclude that the image of P under any B_r-homomorphism from $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, k)$ to $E^{\otimes r}$ must be a scalar multiple of $u^{\otimes r/2}$. □

The symplectic Schur coalgebra is $A_0(n,r) = O_{\Lambda_0^+(m,r)}(k[\mathrm{Sp}_n])$ and $S_0(n,r) = A_0(n,r)^*$, where the left action of Sp_n on $k[\mathrm{Sp}_n]$ comes from right multiplication in Sp_n; see [13] for the generalities. Recall that $E^{\otimes r}$ is self-dual as an Sp_n-module and as a B_r-module. It follows that

$$f_0(A_0(n,r)) = \mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, S_0(n,r)^*) \cong \mathrm{Hom}_{\mathrm{Sp}_n}(S_0(n,r), E^{\otimes r}) \cong E^{\otimes r} \quad \text{and}$$

$$g_0(E^{\otimes r}) = E^{\otimes r} \otimes_{B_r} E^{\otimes r} \cong \mathrm{End}_{B_r}(E^{\otimes r})^* = S_0(n,r)^* \cong A_0(n,r). \tag{6}$$

In the proposition below g denotes the inverse Schur functor from $\mathrm{mod}(k\mathrm{Sym}_t)$ to $\mathrm{mod}(S(n,t))$.

Proposition 2.2

(i) *If $n = 0$ in k and $t = 0$, assume $r \geq 4$. Then we have*

$$g_0(\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V) \cong g(k_{\mathrm{sg}} \otimes V)$$

as Sp_n-modules, for every $k\mathrm{Sym}_t$-module V.

(ii) *Let $\lambda \in \Lambda_0^+(m,r)$. If $\lambda = \emptyset$ and $n = 0$ in k, then assume $r \geq 4$. Then $g_0(\widetilde{\mathcal{M}}(\lambda)) \cong S^\lambda E$ and if* char $k \neq 2$, *then $g_0(\mathcal{M}(\lambda)) \cong \bigwedge^\lambda E$.*

(iii) *Let $\lambda \in \Lambda_0^+(m,r)$. The Sp_n-module $S^\lambda E$ has a unique indecomposable summand $J(\lambda)$ in which $\nabla_0(\lambda)$ has filtration multiplicity > 0 and this multiplicity is equal to* 1. *If* char $k \neq 2$, *then every summand of $\widetilde{\mathcal{M}}(\lambda)$ has a twisted Specht filtration and $f_0(J(\lambda)) \cong \widetilde{\mathcal{Y}}(\lambda)$.*

Proof (i) Since $\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V \cong \big(\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes k_{\mathrm{sg}}\big) \otimes_{k\mathrm{Sym}_t} V$ which is isomorphic to $\mathrm{Hom}_{\mathrm{Sp}_n}(E^{\otimes r}, E^{\otimes t}) \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes V\big)$, this follows from Lemmas 1.1 and 2.4 applied to $E^{\otimes t}$.

(ii) By (i) (with $t = |\lambda|$) we have $g_0(\widetilde{\mathcal{M}}(\lambda)) \cong g(M(\lambda))$ and $g_0(\mathcal{M}(\lambda)) \cong g(k_{\mathrm{sg}} \otimes M(\lambda))$. One easily verifies that $g(M(\lambda)) \cong S^\lambda E$ and, in case char $k \neq 2$, $g(k_{\mathrm{sg}} \otimes M(\lambda)) \cong \bigwedge^\lambda E$.

(iii) Put $t = |\lambda|$. Then $r - t = 2s$ is even. The filtration multiplicity of $\nabla(\lambda)$ in $S^\lambda E$ is 1 and if $\nabla_0(\nu)$ has filtration multiplicity > 0 in $\nabla(\mu)$, then either $\nu = \mu$ and the multiplicity is 1 or $|\nu| < |\mu|$ as one can easily deduce from Lemma 2.2(ii). We conclude that the filtration multiplicity of $\nabla_0(\lambda)$ in $S^\lambda E$ is 1. A direct summand of a module with a good filtration has a good filtration. So, by the Krull–Schmidt theorem, there is a unique indecomposable summand $J(\lambda)$ in which $\nabla_0(\lambda)$ has filtration multiplicity > 0. This proves the first assertion. Now assume char $k \neq 2$. If $\lambda = \emptyset$, then $S^\lambda E = k$, $Z_{r/2} = I_{r/2}$ and $\widetilde{\mathcal{S}}(\lambda) = \widetilde{\mathcal{M}}(\lambda) = \widetilde{\mathcal{Y}}(\lambda) = I_{r/2}$ by [27, Cor. 3.2] and the assertion is obvious. Now assume $\lambda \neq \emptyset$. By (ii) and Theorem 2.1(i) we have $g_0(f_0(M)) \cong M$ canonically for every direct summand of $S^\lambda E$ and $f_0(g_0(V)) \cong V$ canonically for every direct summand of $\widetilde{\mathcal{M}}(\lambda)$. In particular, every direct summand of $\widetilde{\mathcal{M}}(\lambda)$ has a twisted Specht filtration.

Let $I(\lambda) \subseteq S^\lambda E$ be the $S(n,t)$-injective hull of $\nabla(\lambda)$. By [18, 3.6] we have $f(I(\lambda)) = Y(\lambda)$. Put $\pi = \pi_t = \{\mu \in \Lambda_0^+(m,r) \,|\, |\mu| < t\}$. By Remarks 2.1, 1 and 2 we have $f_0(I(\lambda)/O_\pi(I(\lambda))) \cong Z_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes Y(\lambda)\big)$. By [27, Prop. 3.1] $Z_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes Y(\lambda)\big)$ is indecomposable. Since $I(\lambda)/O_\pi(I(\lambda))$ has a good filtration, it must also be indecomposable. Now write $I(\lambda) = \bigoplus_{i=1}^l J_i$ with each J_i an indecomposable Sp_n-module. Then $I(\lambda)/O_\pi(I(\lambda)) \cong \bigoplus_{i=1}^l J_i/O_\pi(J_i)$. So there is a unique j such that $J_j/O_\pi(J_j) \cong I(\lambda)/O_\pi(I(\lambda))$ and $J_i \subseteq O_\pi(I(\lambda))$ for all $i \neq j$. Clearly we must have $J_j \cong J(\lambda)$. Furthermore, since the kernel of $J(\lambda) \to J(\lambda)/O_\pi(J(\lambda))$ has a good filtration, we have that $f_0(J(\lambda))$ surjects onto $Z_s \otimes_{k\mathrm{Sym}_t} \big(k_{\mathrm{sg}} \otimes Y(\lambda)\big)$. So $f_0(J(\lambda)) \cong \widetilde{\mathcal{Y}}(\lambda)$. □

Remarks 2.2 1. Drop the assumption that $\delta = -n$. Assume that char $k \neq 2, 3$ and also $r \neq 2, 4$ in case $\delta = 0$. Then it follows from [24, Thm 1.6] and the results in [27] that a direct summand of a module with a Specht or twisted Specht filtration has again such a filtration. In particular this applies to the Young modules for the Brauer algebra and their twisted versions.
2. The class of $S_0(n,r)$-modules M for which $g_0(f_0(M)) \cong M$ canonically, is closed under taking direct summands and direct sums. In particular it contains the injective $S_0(n,r)$-modules, since, by (6), it contains $A_0(n,r)$. For the same reason the class of B_r-modules V for which $f_0(g_0(V)) \cong V$ canonically, contains the projective B_r-modules.
3. For $\lambda \in X_0$ let I_λ be the λ-weight space of $A_0(n,r)$ for the action of T_0 which comes from left multiplication in G. One shows as in [14, 2.4] that $\mathrm{Hom}(-, I_\lambda) \cong M \mapsto (M_\lambda)^*$ as functors on $\mathrm{mod}(S_0(n,r))$. In particular, I_λ is an injective $S_0(n,r)$-module. Furthermore, $A_0(n,r) = \bigoplus_{\lambda \in \Lambda_0(m,r)} I_\lambda$ and $I_\lambda \cong I_{w(\lambda)}$ for every w in the Weyl group W. So every injective $S_0(n,r)$-module is a direct sum of direct summands of I_λ's, $\lambda \in \Lambda_0^+(m,r)$. Since the nondegenerate bilinear form of $E^{\otimes r}$ is also nondegenerate on the weight spaces of $E^{\otimes r}$ for T_0, we have that those weight spaces are self-dual B_r-modules. So $f_0(I_\lambda) = (E^{\otimes r})_{\lambda,0}$, the λ-weight space of $E^{\otimes r}$ for T_0. Note that $g_0((E^{\otimes r})_{\lambda,0}) = I_\lambda$ by the preceding remark. By Lemma 2.2(i) we have $(E^{\otimes r})_{\lambda,0} \cong M(\lambda)$ if λ is a partition of r. Here the diagrams with a horizontal edge act as 0 on $M(\lambda)$. By [36, Thm 9.51] (with $S = C = k$), we have

$$L^i g_0(V) = \mathrm{Tor}_i^{B_r}(E^{\otimes r}, V) \cong \mathrm{Ext}^i_{B_r}(V, E^{\otimes r})^*$$

as vector spaces, for every B_r-module V. Here $L^i g_0$ denotes the ith left derived functor of g_0. So to show that, under certain conditions on m, r and the field k, $L^1 g_0(V) = 0$ whenever V has a twisted Specht filtration, it suffices to show that $\mathrm{Ext}^1_{B_r}(\widetilde{\mathcal{S}}(\lambda), (E^{\otimes r})_{\mu,0}) = 0$ for all $\lambda, \mu \in \Lambda_0^+(m,r)$. This would mean that g_0 maps exact sequences of modules with a twisted Specht filtration to exact sequences and it would imply as in the proof of [20, Prop. 10.6] that $g_0(\widetilde{\mathcal{S}}(\lambda)) = \nabla_0(\lambda)$. This, in turn, would imply that g_0 maps B_r-modules with a twisted Specht filtration to Sp_n-modules with a good filtration.

3 The Jantzen sum formula and a block result for $S_0(n, r)$

The notation is as in the previous section. In this section we assume that the field k is of positive characteristic $p > 2$. Let u be the unique integer satisfying $-p/2 < n - up < p/2$.

Put

$$\delta = -(n - up).$$

Throughout this section we will be working with the root system of type C_m in the vector space $\mathbb{R}^m$ endowed with the standard inner product

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i.$$

The weight lattice is identified with $\mathbb{Z}^m$. The positive roots are $2\varepsilon_i$, $1 \le i \le m$, and $\varepsilon_i \pm \varepsilon_j$, $1 \le i < j \le m$. Recall that the Weyl group $W(C_m)$ acts by signed permutations and that the *dot action* of $W(C_m)$ on $\mathbb{R}^m$ is given by

$$w \cdot x = w(x + \rho) - \rho,$$

where

$$\rho = (m, \dots, 2, 1).$$

Note that $\mathbb{Z}^m$ is stable under the dot action.

Let G be a reductive group and let X be the group of weights relative to a fixed maximal torus. Jantzen has defined for every Weyl module $\Delta(\lambda)$ of G a descending filtration $\Delta(\lambda) = \Delta(\lambda)^0 \supseteq \Delta(\lambda)^1 \supseteq \cdots$ such that $\Delta(\lambda)/\Delta(\lambda)^1 \cong L(\lambda)$ and $\Delta(\lambda)^i = 0$ for i big enough. The Jantzen sum formula [29, II.8.19] relates the formal characters of the $\Delta(\lambda)^i$ with the Weyl characters $\chi(\mu)$:

$$\sum_{i>0} \operatorname{ch} \Delta(\lambda)^i = \sum \nu_p(lp)\chi(s_{\alpha,l} \cdot \lambda), \tag{7}$$

where the sum on the right is over all pairs (α, l), with l an integer ≥ 1 and α a positive root such that $\langle \lambda + \rho, \alpha^\vee\rangle - lp > 0$. Here ν_p is the p-adic valuation, $\alpha^\vee = \frac{2}{\langle\alpha,\alpha\rangle}\alpha$ and $s_{\alpha,l}$ is the affine reflection of $\mathbb{R} \otimes_{\mathbb{Z}} X$ defined by $s_{\alpha,l}(x) = x - a\alpha$, where $a = \langle x, \alpha^\vee\rangle - lp$. It should be noted that the $s_{\alpha,l} \cdot \lambda$ are in general not dominant. But if $\chi(s_{\alpha,l} \cdot \lambda) \ne 0$, then it can be written as $\pm\chi(\mu)$ for some dominant weight μ using [29, II.5.9(1)]. The $\chi(\mu)$, μ a dominant weight, form a $\mathbb{Z}$-basis of $(\mathbb{Z}X)^W$, where W denotes the Weyl group.

We are now going to prove a strengthened version of the Jantzen sum formula for the symplectic group in a certain generic situation and deduce from this formula a block result for $S_0(n, r)$. In Sect. 5 we will then deduce from this the block result [10, Thm 4.2] for the Brauer algebra in characteristic 0. Following Cox, De Visscher and Martin we define $\hat{\rho} \in \frac{1}{2}\mathbb{Z}^r$ by

$$\hat{\rho} = \left(-\frac{\delta}{2}, -\frac{\delta}{2} - 1, \dots, -\frac{\delta}{2} - (r-1)\right)$$

and the *star action* (called "dot action" in [10]) of $W(C_r)$ on $\mathbb{R}^r$ by

$$w \star x = w(x + \hat{\rho}) - \hat{\rho}.$$

Note again that $\mathbb{Z}^r$ is stable under this action. The Weyl group $W(C_r)$ contains the Weyl group $W(D_r)$ of type D_r which consists of those signed permutations that involve an even number of sign changes. When $m \ge r$ we will use the convention that r-tuples can also be considered as m-tuples by extending them with zeros. For a partition λ, we denote its length by $l(\lambda)$ and we denote the transposed partition by λ'.

Theorem 3.1 *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$. If $n + 2r < p$, then $\Delta_0(\lambda)$ is irreducible. If $|\delta| + 2r < p/2$ and $u \neq 0$, then $m > r$ and we have*

$$\sum_{i>0} \mathrm{ch}\, \Delta_0(\lambda)^i = \nu_p(up) \sum \chi_0(s_\alpha \star \lambda), \tag{8}$$

where the sum on the right is over all positive roots $\alpha = \varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq r$, with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$.

Proof The first assertion follows immediately from the fact that then the sum on the right in (7) is empty. Now we assume that $|\delta| + 2r < p/2$ and $u \neq 0$. Clearly $m > r$. The idea is to deduce (8) from (7) by showing that certain summands on the right-hand side of (7) may be omitted. In the proof below we will need some extra notation. For $i \in \{1, \ldots, m\}$, we put $i' = m + 1 - i$. Note that this differs from the notation in Sect. 1.1. For $x \in \mathbb{R}^m$ we define $\breve{x}$ to be the reversed tuple of x. So $\breve{x}_i = x_{i'}$. Note that $\breve{\rho}_i = i$ and that the first $m - r$ entries of $(\lambda + \rho)\breve{}$ form the interval $\{1, 2 \ldots, m - r\}$. Furthermore, α will always denote a positive root in the root system of type C_m. We will use the following fact. Let $\lambda \in \mathbb{Z}^m$. Then

$$\begin{aligned} &\chi_0(\lambda) \neq 0 \quad \text{if and only if,} \\ &(\lambda + \rho)_i \neq 0 \quad \text{for all } i \in \{1, \ldots, m\}, \quad \text{and} \\ &(\lambda + \rho)_i \neq \pm(\lambda + \rho)_j \quad \text{for all } i, j \in \{1, \ldots, m\} \text{ with } i \neq j. \end{aligned}$$

The proof will consist of three lemma's.

Lemma 1 *Assume $\alpha = \varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq m$ and $\langle \lambda + \rho, \alpha^\vee \rangle = a + lp$, $a, l > 0$. Then $\chi_0(s_{\alpha,l} \cdot \lambda) = 0$.*

Proof First we redefine i and j by replacing (i, j) by (j', i'). So $1 \leq i < j \leq m$ and $\alpha = \alpha^\vee = \varepsilon_{j'} - \varepsilon_{i'}$. We have $\langle \lambda + \rho, \alpha^\vee \rangle = \breve{\lambda}_j + j - (\breve{\lambda}_i + i)$ and $s_{\alpha,l} \cdot \lambda = \lambda - a\alpha$. So $s_{\alpha,l}(\lambda + \rho)\breve{}_i = \breve{\lambda}_i + i + a$ and $s_{\alpha,l}(\lambda + \rho)\breve{}_j = \breve{\lambda}_j + j - a$.

We have $a \leq \langle \lambda + \rho, \alpha^\vee \rangle - p \leq m + r - (\breve{\lambda}_i + i) - p$. So $\breve{\lambda}_i + i + a \leq m + r - p < m - r$, since $2r < p$. It follows that $i < m - r$ and $\breve{\lambda}_i = 0$. Now $i < i + a < \breve{\lambda}_j + j$ and $i + a < m - r$. So $s_{\alpha,l} \cdot \lambda + \rho = s_{\alpha,l}(\lambda + \rho)$ contains a repeat and $\chi_0(s_{\alpha,l} \cdot \lambda) = 0$. □

Lemma 2 *Let Φ_1 be the set of roots $\varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq m$, $j > r$ and let Φ_2 be the set of roots $2\varepsilon_i$, $1 \leq i \leq m$. Furthermore, let S_1 be the set of pairs (α, l) such that $\alpha \in \Phi_1$, l an integer ≥ 1, $\langle \lambda + \rho, \alpha^\vee \rangle - lp > 0$ and $\chi_0(s_{\alpha,l} \cdot \lambda) \neq 0$, and let S_2 be the corresponding set for Φ_2. Then there exists a map $\varphi : S_1 \to \Phi_2$ such that:*

(i) $(\alpha, l) \mapsto (\varphi(\alpha, l), l)$ *is a bijection from S_1 onto S_2.*
(ii) $\chi_0(s_{\alpha,l} \cdot \lambda) = -\chi_0(s_{\varphi(\alpha,l),l} \cdot \lambda)$.

Proof Let $(\alpha, l) \in S_1$. Write $\alpha = \alpha^\vee = \varepsilon_{j'} + \varepsilon_{i'}$, $1 \leq i < j \leq m$, $i \leq m - r$. Put $a = \langle \lambda + \rho, \alpha^\vee \rangle - lp$. We have $\breve{\lambda}_i = 0$, $\langle \lambda + \rho, \alpha^\vee \rangle = \breve{\lambda}_j + j + i$ and $s_{\alpha,l} \cdot \lambda = \lambda - a\alpha$. So $s_{\alpha,l}(\lambda + \rho)\breve{}_i = i - a$ and $s_{\alpha,l}(\lambda + \rho)\breve{}_j = \breve{\lambda}_j + j - a$.

We have $i - a < i < \breve{\lambda}_j + j$. So, if $i - a \geq 0$, then $s_{\alpha,l}(\lambda + \rho)$ contains a repeat or a zero. Therefore $a - i > 0$. We have $a \leq \langle \lambda + \rho, \alpha^\vee \rangle - p \leq m + r + i - p < m - r + i$, since $2r < p$. So $0 < a - i \leq m - r$. Now $a - i < \breve{\lambda}_j + j$. So if $a - i \neq i$, then $s_{\alpha,l}(\lambda + \rho)$ contains a repeat up to sign. Therefore $a = 2i$ and $s_{\alpha,l}(\lambda + \rho)\breve{}_i = -i$. Now put $\varphi(l, \alpha) = 2\varepsilon_{j'}$. Note that $(2\varepsilon_{j'})^\vee = \varepsilon_{j'}$. So $\langle \lambda + \rho, \varphi(l, \alpha)^\vee \rangle = \breve{\lambda}_j + j = i + lp$. Furthermore, $s_{\alpha,l}(\lambda + \rho)$ is obtained from $s_{\varphi(l,\alpha),l}(\lambda + \rho)$ by changing the sign of the ith coordinate. This proves (ii) and that $(\alpha, l) \mapsto (\varphi(\alpha, l), l)$ is an injection from S_1 to S_2.

Now let $(\beta, l) \in S_2$. Write $\beta = 2\varepsilon_{j'}, 1 \le j \le m$. Define i by the equation $\breve{\lambda}_j + j = i + lp$. Clearly $i > 0$. Furthermore, $i \le \breve{\lambda}_j + j - p < j$, since $r < p$ and $i \le \breve{\lambda}_j + j - p < m - r$, since $2r < p$. Put $\alpha = \alpha^\vee = \varepsilon_{j'} + \varepsilon_{i'}$. From the previous computations it now follows that $(\alpha, l) \in S_1$ and it is clear that $\varphi(\alpha, l) = \beta$. This proves (i). □

Lemma 3 *Assume* $\alpha = \varepsilon_i + \varepsilon_j$, $1 \le i < j \le r$, $\langle \lambda + \rho, \alpha^\vee \rangle = a + lp$, $a, l > 0$ *and* $\chi_0(s_{\alpha,l} \cdot \lambda) \ne 0$. *Then* $l = u$. *Furthermore, the entries of* $s_{\alpha,u}(\lambda + \rho)$ *are distinct and strictly positive.*

Proof First we redefine i and j by replacing (i, j) by (j', i'). So $m - r < i < j \le m$ and $\alpha = \alpha^\vee = \varepsilon_{j'} + \varepsilon_{i'}$. Note that the first $m - r$ entries of $s_{\alpha,l}(\lambda + \rho)\breve{}$ form the interval $\{1, 2 \ldots, m - r\}$. We have $0 < a < \langle \lambda + \rho, \alpha^\vee \rangle = \breve{\lambda}_j + j + \breve{\lambda}_i + i = a + lp$. Clearly $\breve{\lambda}_i + i - a = s_{\alpha,l}(\lambda + \rho)\breve{}_i \ne 0$. We have $a \le \breve{\lambda}_j + j + \breve{\lambda}_i + i - p \le m + r + \breve{\lambda}_i + i - p < m - r + \breve{\lambda}_i + i$, since $2r < p$. So if $\breve{\lambda}_i + i - a < 0$, then $0 < a - \breve{\lambda}_i - i < m - r$ and $s_{\alpha,l}(\lambda + \rho)\breve{}$ would contain a repeat up to sign. So we have $\breve{\lambda}_i + i - a > 0$. Since $\breve{\lambda}_j + j - a > \breve{\lambda}_i + i - a$ this shows that all entries of $s_{\alpha,l}(\lambda + \rho)$ are (distinct and) strictly positive. If $\breve{\lambda}_i + i - a \le m - r$, then $s_{\alpha,l}(\lambda + \rho)\breve{}$ would contain a repeat. So we have $\breve{\lambda}_i + i - a > m - r$. It follows that $a < \breve{\lambda}_i + i - m + r < m + r - (m - r) = 2r$. So $0 < \langle \lambda + \rho, \alpha^\vee \rangle - lp = a < 2r < p/2$. On the other hand $-\delta - 2r = n - 2r - up < \langle \lambda + \rho, \alpha^\vee \rangle - up < n + 2r - up = -\delta + 2r$. Since $|\delta| + 2r < p/2$, this implies that $u = l$. □

We can now finish the proof of (8). By Lemmas 1 and 2 we can restrict the sum on the right in (7) to positive roots $\alpha = \varepsilon_i + \varepsilon_j$, $1 \le i < j \le r$ with $\langle \lambda + \rho, \alpha^\vee \rangle - lp > 0$. Assume now that for such a root α we have $\chi_0(s_{\alpha,l} \cdot \lambda) \ne 0$. Then we have $l = u$, by Lemma 3. Now $\hat{\rho}_i = \rho_i - (u/2)p$, so $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle = \lambda_i + \rho_i + \lambda_j + \rho_j - up = \langle \lambda + \rho, \alpha^\vee \rangle - lp > 0$. So $s_{\alpha,u} \cdot \lambda = s_\alpha \star \lambda$. On the other hand, if $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$ and $\chi_0(s_\alpha \star \lambda) \ne 0$, then (α, u) gives the same nonzero summand in the sum on the right of (7). □

Corollary *Assume that* $|\delta| + 2r < p/2$ *and that* $\delta \ge 2r - 1$. *Then, under the above assumptions,* $\Delta_0(\lambda)$ *is irreducible for any partition* λ *with* $l(\lambda), l(\lambda') \le r$.

Proof Clearly we may assume that $u \ne 0$. The assertion now follows immediately from (8) and the fact that $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \le -\delta + 2r - 1$. □

Remark Note that for the proofs of Lemmas 1 and 2 we only needed that $r < 2p$ and no assumptions on δ.

We now turn our attention to the blocks of the symplectic Schur algebra. Let S be a finite dimensional algebra. Fix a labelling set X for the isomorphism classes of irreducible S-modules. For $\lambda \in X$, let $L(\lambda)$ be the corresponding irreducible S-module. On X the *block relation* is defined as the smallest equivalence relation $\sim$ such that $\mathrm{Ext}^1_S(L(\lambda), L(\mu)) \ne 0$ implies $\lambda \sim \mu$. When $\lambda \sim \mu$, we say that $L(\lambda)$ and $L(\mu)$ (or just λ and μ) are in the same block. If S is quasi-hereditary and if for all λ, $\Delta(\lambda)$ and $\nabla(\lambda)$ have the same composition factors, then $\sim$ is the smallest equivalence relation such that $[\Delta(\lambda) : L(\mu)] \ne 0$ implies $\lambda \sim \mu$. Here we denote for an S-module M the multiplicity of $L(\mu)$ as a composition factor of M by $[M : L(\mu)]$.

In the remainder of this section we assume that $m \ge r$. To prove the block result, we need some more notation. For $x, y \in \mathbb{R}^r$ we write $x \subseteq y$ if $x_i \le y_i$ for all $i \in \{1, \ldots, r\}$. Furthermore, we define $x \cap y = \min(x_i, y_i)_{i \in \{1,\ldots,r\}}$. Note that $x \cap y$ is (strictly) decreasing if this holds for x and y. We have $(x + z) \cap (y + z) = (x \cap y) + z$ for any $z \in \mathbb{R}^r$. The next lemma is suggested by the proof of [9, Thm. 4.2]: there it is shown that on $\Lambda_0^+(m, r)$ conjugacy under the star action is equivalent to "balancedness" which clearly has the property stated in (ii).

Proposition 3.1 *Let* $\lambda, \mu \in \mathbb{R}^r$.

(i) *Assume* λ *and* μ *are strictly decreasing. If they are conjugate under the action of* $W(C_r)$, *then the same holds for* λ *and* $\lambda \cap \mu$.
(ii) *Assume* λ *and* μ *are decreasing. If they are conjugate under the star action of* $W(C_r)$, *then the same holds for* λ *and* $\lambda \cap \mu$.
(iii) *Assertions* (i) *and* (ii) *also hold with* $W(C_r)$ *replaced by* $W(D_r)$.

Proof Clearly (ii) follows from (i) and the second assertion of (iii) follows from the first.

Assume (i) holds and that λ and μ are strictly decreasing. We will show that then the first assertion of (iii) holds. If λ contains a zero, then our result follows immediately from (i), so assume that this is not the case. For $x \in \mathbb{R}^r$ denote the number of entries < 0 by $N_-(x)$. Then we have for $x, y \in (\mathbb{R}\backslash\{0\})^r$ that $N_-(x \cap y) = \max\{N_-(x), N_-(y)\}$. Furthermore we have that x and y are conjugate under the action of $W(D_r)$ if and only if they are conjugate under the action of $W(C_r)$ and $N_-(x) \equiv N_-(y) \pmod 2$. The result now follows.

It remains to prove (i). Assume λ and μ are strictly decreasing and that they are conjugate under the action of $W(C_r)$. By the pigeonhole principle it suffices to show for each real number a the following.

1. If a and $-a$ occur in λ (this includes the case that $a = 0$ occurs in λ), then they occur in $\lambda \cap \mu$.
2. If $a > 0$ and a occurs in λ, but $-a$ does not occur in λ, then a or $-a$ occurs in $\lambda \cap \mu$.

Let $a \in \mathbb{R}$. If a and $-a$ occur in λ, then a and $-a$ occur in μ. So for 1 it suffices to show that if a occurs in λ and μ, then it occurs in $\lambda \cap \mu$. Let i be the index with $\lambda_i = a$. If $\min\{\lambda_i, \mu_i\} = \lambda_i$, then we are done, so assume $\mu_i < \lambda_i = a$. Then a occurs in μ before position i, so $\mu_j = a$ for some $j < i$. But then $\lambda_j > \lambda_i = a = \mu_j$ and $\min\{\lambda_j, \mu_j\} = a$.

Now assume that $a > 0$ and a occurs in λ, but $-a$ does not occur in λ. Because of the above we may assume that a does not occur in μ. Then $-a$ occurs in μ. For a contradiction, assume that neither a nor $-a$ occurs in $\lambda \cap \mu$. Let i_0 be the index with $\lambda_{i_0} = a$ and let j_0 be the index with $\mu_{j_0} = -a$. Then we have $b := \mu_{i_0} < a$ and $c := \lambda_{j_0} < -a$. If $j_0 \le i_0$, then we would have $-a > c = \lambda_{j_0} \ge \lambda_{i_0} = a$, a contradiction. So $j_0 > i_0$. In a picture:

$$
\begin{array}{cccccc}
 & & i_0 & & j_0 & \\
\lambda & \cdots & a & \cdots & c & \cdots \\
 & & \vee & & \wedge & \\
\mu & \cdots & b & \cdots & -a & \cdots
\end{array}
$$

Now we define recursively a sequence of numbers $\sigma_0, \sigma_1, \sigma_2, \ldots \in \{1, \ldots, r\}$ as follows. $\sigma_0 = j_0$. Assume that $k \ge 0$ and that σ_k is defined. If λ_{σ_k} occurs in μ, then define σ_{k+1} as the index with $\mu_{\sigma_{k+1}} = \lambda_{\sigma_k}$, otherwise we define it as the index with $\mu_{\sigma_{k+1}} = -\lambda_{\sigma_k}$. To reach a contradiction it is clearly sufficient to show that:

(1) $\sigma_i < i_0$ or $\sigma_i > j_0$ for all $i > 0$ and
(2) $\sigma_i \ne \sigma_j$ for $i \ne j$.

We do this by induction. Let $k \ge 0$ and assume that $\sigma_0, \ldots, \sigma_k$ are distinct and satisfy (1). If $\sigma_k \ge j_0$, then $\lambda_{\sigma_k} \le \lambda_{j_0} = c < -a = \mu_{j_0}$ and $-\lambda_{\sigma_k} > a > b = \mu_{i_0}$. If $\sigma_k < i_0$, then $\lambda_{\sigma_k} > \lambda_{i_0} = a > b = \mu_{i_0}$ and $-\lambda_{\sigma_k} < -a = \mu_{j_0}$. So, by the definition of σ_{k+1}, either $\sigma_{k+1} < i_0$ or $\sigma_{k+1} > j_0$.

Finally we have to show that $\sigma_{k+1} \ne \sigma_j$ for $j \le k$. Assume, again for a contradiction, that $\sigma_{k+1} = \sigma_i$ for some $i \le k$. Because of what we just proved we have $i > 0$. We have $\pm\lambda_{\sigma_k} = \mu_{\sigma_{k+1}} = \mu_{\sigma_i} = \pm\lambda_{\sigma_{i-1}}$. Because of the induction hypothesis we have $\sigma_k \ne \sigma_{i-1}$, so $\lambda_{\sigma_k} = -\lambda_{\sigma_{i-1}}$. Since μ is a signed permutation of λ, this means that λ_{σ_k} and $\lambda_{\sigma_{i-1}}$ occur in μ. But then, by the definition of μ, $\mu_{\sigma_{k+1}} = \lambda_{\sigma_k}$ and $\mu_{\sigma_i} = \lambda_{\sigma_{i-1}}$. So $\lambda_{\sigma_k} = \lambda_{\sigma_{i-1}}$, a contradiction. $\square$

Remark We haven't checked whether (i) also holds for not necessarily strictly decreasing λ and μ. We certainly don't need this more general result.

From now on we assume that $|\delta| + 2r < p/2$. For $x \in \mathbb{R}^r$, we denote by $\mathrm{sort}(x)$, the r-tuple which is obtained by sorting x in descending order.

Lemma 3.1 *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$ and let $\alpha = \varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq r$, be a positive root with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$ and $\chi_0(s_\alpha \star \lambda) \neq 0$. Put $\mu = \mathrm{sort}(s_\alpha(\lambda + \hat{\rho})) - \hat{\rho}$. Then all entries of $s_\alpha(\lambda + \hat{\rho})$ are distinct and μ is a partition with $\mu \subsetneqq \lambda$. Let w be the permutation such that $w(s_\alpha(\lambda + \hat{\rho}))$ is (strictly) decreasing. Then $\mu = w \star s_\alpha \star \lambda$ and $\chi_0(s_\alpha \star \lambda) = \mathrm{sgn}(w)\chi_0(\mu)$.*

Proof We have $s_\alpha \star \lambda = s_{\alpha,up} \cdot \lambda$. Since $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$, we have that $s_\alpha(\lambda + \hat{\rho}) \subseteq \lambda + \hat{\rho}$ and the same holds for the r-tuple obtained from $s_\alpha(\lambda + \hat{\rho})$ by sorting it in descending order. So $\mu \subseteq \lambda$. We have $|\mu| = |s_\alpha \star \lambda| < |\lambda|$, so $\mu \neq \lambda$. Recall that the final $m - r$ entries of $s_{\alpha,lp}(\lambda + \rho)$ form the reversed interval $\{m - r, \dots, 2, 1\}$. Furthermore, the entries of $s_{\alpha,lp}(\lambda + \rho) = s_\alpha(\lambda + \hat{\rho}) + \rho - \hat{\rho}$ are distinct and strictly positive by Lemma 3 in the proof of Theorem 3.1. So sorting them in descending order only involves the first r entries. The result now follows from [29, II.5.9(1)] and the fact that $\rho - \hat{\rho}$ has the same value $\frac{u}{2}p$ on the first r entries. $\square$

We can now deduce from Theorem 3.1 a linkage principle for the symplectic Schur algebra.

Lemma 3.2 *Let λ and μ be partitions with $l(\lambda), l(\lambda'), l(\mu), l(\mu') \leq r$.*

- (i) *If $[\Delta_0(\lambda) : L_0(\mu)] \neq 0$, then $\mu \subseteq \lambda$ and λ and μ are conjugate under the star action of $W(D_r)$.*
- (ii) *If $\lambda, \mu \in \Lambda_0^+(m, r)$ are in the same block of $S_0(n, r)$, then they are conjugate under the star action of $W(D_r)$.*

Proof Assertion (ii) follows immediately from (i). We will show (i) by induction on λ with respect to the ordering $\subseteq$. If $\mu = \lambda$, then there is nothing to prove. So assume that $\mu \neq \lambda$. Then $\Delta_0(\lambda)$ is not irreducible and we must have $u \neq 0$. Furthermore, $L_0(\mu)$ is a composition factor of $\bigoplus_{i>0} \Delta_0(\lambda)^i$. Since the $\mathrm{ch}\, L_0(\nu)$'s form a basis of $(\mathbb{Z}X_0)^{W(C_m)}$, we must have that $\mathrm{ch}\, L_0(\mu)$ occurs in some $\chi_0(s_\alpha \star \lambda)$ occurring on the right in (8). By Lemma 3.1 we have that $\mathrm{ch}\, L_0(\mu)$ occurs in $\chi_0(\nu)$ for some partition ν with $\nu \subsetneqq \lambda$ and $\nu \in W(D_r) \star \lambda$. We can now finish by applying the induction hypothesis. $\square$

The next lemma shows that there are no repetitions or cancellations in the sum on the right in (8).

Lemma 3.3 *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$ and let $\alpha = \varepsilon_i + \varepsilon_j$ and $\beta = \varepsilon_k + \varepsilon_l$, $1 \leq i < j \leq r$ and $1 \leq k < l \leq r$, be positive roots with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$, $\langle \lambda + \hat{\rho}, \beta^\vee \rangle > 0$, $\chi_0(s_\alpha \star \lambda) \neq 0$ and $\chi_0(s_\beta \star \lambda) \neq 0$. If $\alpha \neq \beta$, then $\chi_0(s_\alpha \star \lambda) \neq \pm\chi_0(s_\beta \star \lambda)$.*

Proof Put $\mu_\alpha = \mathrm{sort}(s_\alpha(\lambda + \hat{\rho})) - \hat{\rho}$ and $\mu_\beta = \mathrm{sort}(s_\beta(\lambda + \hat{\rho})) - \hat{\rho}$. First assume that $i \neq k$, say $i < k$. The we have that the ith value of $\mathrm{sort}(s_\beta(\lambda + \hat{\rho}))$ is $(\lambda + \hat{\rho})_i$, but the ith value of $\mathrm{sort}(s_\alpha(\lambda + \hat{\rho}))$ is strictly less than this value. So $\mu_\alpha \neq \mu_\beta$.

Now assume that $i = k$ and that $j \neq l$. We have $s_\gamma \star x = x - \langle x + \hat{\rho}, \gamma^\vee \rangle \gamma$. So for the coordinate sum $|x|$ of x we have $|s_\gamma \star x| = |x| - 2\langle x + \hat{\rho}, \gamma^\vee \rangle$ if γ is a short root. Now $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \neq \langle \lambda + \hat{\rho}, \beta^\vee \rangle$ and we always have $|\mathrm{sort}(x)| = |x|$, so μ_α and μ_β are partitions of different numbers. □

Corollary *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$ and let Λ_λ be the set of partitions ν such that $\nu = \mathrm{sort}(s_\alpha(\lambda + \hat{\rho})) - \hat{\rho}$ for some positive root $\alpha = \varepsilon_i + \varepsilon_j$ with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$ and $\chi_0(s_\alpha \star \lambda) \neq 0$. Assume that $\Lambda_\lambda \neq \emptyset$ and let μ be an $\subseteq$-maximal element of Λ_λ. Then $[\Delta_0(\lambda) : L_0(\mu)] \neq 0$.*

Proof If $\mathrm{ch}(L_0(\mu))$ occurs in $\chi_0(\nu)$ for some $\nu \in \Lambda_\lambda$, then $\mu \subseteq \nu$ by Lemma 3.2 and $\nu = \mu$ by the maximality of μ. So in the sum in (8) only $\pm\chi_0(\mu)$ contains $\mathrm{ch}(L_0(\mu))$ and $\chi_0(\mu)$ must appear with positive coefficient. □

Lemma 3.4 *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$ and let $\alpha = \varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq r$, be a positive root with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$. Then $\chi_0(s_\alpha \star \lambda) \neq 0$ if and only if $(\lambda + \hat{\rho})_i, (\lambda + \hat{\rho})_j < \frac{\delta}{2} + r$ and $(\lambda + \hat{\rho})_k \neq -(\lambda + \hat{\rho})_i$ and $(\lambda + \hat{\rho})_k \neq -(\lambda + \hat{\rho})_j$ for all $k \in \{1, \ldots, r\} \backslash \{i, j\}$.*

Proof We have $s_\alpha \star \lambda = s_{\alpha,up} \cdot \lambda$. By Lemma 3 in the proof of Theorem 3.1, $\chi_0(s_\alpha \star \lambda) \neq 0$ if and only if $s_{\alpha,up}(\lambda + \rho)$ has no repetitions and all its entries are strictly positive. The latter is the case if and only if $s_\alpha(\lambda + \hat{\rho})$ has no repetitions and all its entries are $> -\frac{\delta}{2} - r$. But the only entries of $s_\alpha(\lambda + \hat{\rho})$ that could be $\leq -\frac{\delta}{2} - r$ are the jth entry $-(\lambda + \hat{\rho})_i$ and the ith entry $-(\lambda + \hat{\rho})_j$. □

Lemma 3.5 *Let λ be a partition with $l(\lambda), l(\lambda') \leq r$. Then $\Delta_0(\lambda)$ is not irreducible if and only if there exists a partition $\mu \subsetneq \lambda$ which is conjugate to λ under the star action of $W(D_r)$.*

Proof First we note that $\Delta_0(\lambda)$ is not irreducible if and only if the sum on the right in (8) is nonzero. If $\Delta_0(\lambda)$ is not irreducible, then a μ as stated must exist by Lemma 3.1. Now assume that such a μ exists. By Lemma 3.3 it suffices to show that there exists a positive root $\alpha = \varepsilon_i + \varepsilon_j$, $1 \leq i < j \leq r$, with $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle > 0$ and $\chi_0(s_\alpha \star \lambda) \neq 0$.

Put $\tilde{\lambda} = \lambda + \hat{\rho}$ and $\tilde{\mu} = \mu + \hat{\rho}$. Let $w \in W(D_r)$ such that $\tilde{\mu} = w(\tilde{\lambda})$. Write $w = w_1 w_2$, where w_1 is a permutation and w_2 changes an even number of signs. Note that w_1 is the unique permutation that sorts $w_2(\tilde{\lambda})$ into descending order. Let $I \subseteq \{1, \ldots, r\}$ be the set of indices whose signs are changed by w_2. For $i \in I$ and $k \in \{1, \ldots, r\} \backslash I$ we have $\tilde{\lambda}_i \neq -\tilde{\lambda}_k$, since all entries of $\tilde{\mu}$ are distinct. If for some $i, j \in I$ with $i \neq j$ we have $\tilde{\lambda}_i = -\tilde{\lambda}_j$ then the sign changes on the ith and jth position cancel each other for $\tilde{\lambda}$. So, after modifying w_1, w_2 and I, we may assume that this does not happen and then we have $\tilde{\lambda}_i \neq -\tilde{\lambda}_k$ for any $i \in I$ and $k \in \{1, \ldots, r\} \backslash \{i\}$. Note that $I \neq \emptyset$, since $\tilde{\mu} \neq \tilde{\lambda}$ and $\tilde{\mu}$ and $\tilde{\lambda}$ are strictly descending. Furthermore $|I|$ is even, since we removed an even number of indices from the original set I.

We have $2\sum_{i \in I} \tilde{\lambda}_i = |\tilde{\lambda} - \tilde{\mu}| = |\lambda - \mu| > 0$. But then there must exist $i, j \in I, i \neq j$, such that $\tilde{\lambda}_i + \tilde{\lambda}_j > 0$. Put $\alpha = \varepsilon_i + \varepsilon_j$. Then $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle = \tilde{\lambda}_i + \tilde{\lambda}_j > 0$. Since all entries of $\hat{\rho}$ are $> -\frac{\delta}{2} - r$, the same holds for the entries of $\tilde{\mu}$. In particular this holds for $-\tilde{\lambda}_i$ and $-\tilde{\lambda}_j$. Since $i, j \in I$, we have that for all $k \in \{1, \ldots, r\} \backslash \{i, j\}$, $\tilde{\lambda}_k \neq -\tilde{\lambda}_i$ and $\tilde{\lambda}_k \neq -\tilde{\lambda}_j$. So, by Lemma 3.4, $\chi_0(s_\alpha \star \lambda) \neq 0$. □

Our proof of the block result for $S_0(n, r)$ is very similar to that for the Brauer algebra in [9, Cor. 6.7].

Theorem 3.2 *Assume that $m \geq r$ and that $|\delta| + 2r < p/2$. Let $\lambda, \mu \in \Lambda_0^+(r, r)$. Then λ and μ are in the same block of $S_0(n, r)$ if and only if they are conjugate under the star action of $W(D_r)$.*

Proof By Lemma 3.2(ii) we only have to show that λ and μ are in the same block of $S_0(n, r)$ if they are conjugate under the star action of $W(D_r)$. By Proposition 3.1(iii) every linkage class under the star action contains a unique $\subseteq$-minimal element. So it suffices to show that if λ is not $\subseteq$-minimal in its linkage class, then it is not $\subseteq$-minimal in its block. This follows immediately from Lemma 3.5 and Lemma 3.2(i) (or the corollary to Lemma 3.3). □

4 The orthogonal Schur algebra and Schur functor

Throughout this section we assume that char $k \neq 2$. Furthermore, n is an integer ≥ 2 and we put $m = \lfloor n/2 \rfloor$. Let $i \mapsto i'$ be the involution of $\{1, \ldots, n\}$ defined by $i' := n + 1 - i$ and define the $n \times n$-matrix J with coefficients in k by $J_{ij} = \delta_{ij'}$. So

$$J = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}.$$

Let $E = k^n$ be the space of column vectors of length n with standard basis $e_1, \ldots, e_n$. On E we define the nondegenerate symmetric bilinear form $(\ ,\ )$ by

$$(u, v) := u^T J v = \sum_{i=1}^{n} u_i v_{i'}.$$

Then $(e_i, e_j) = J_{ij}$. The *orthogonal group* $\mathrm{O}_n = \mathrm{O}_n(k)$ is defined as the group of $n \times n$-matrices over k that satisfy $A^T J A = J$, that is, the invertible matrices for which the corresponding automorphism of E preserves the form $(\ ,\ )$. The *special orthogonal group* $\mathrm{SO}_n = \mathrm{SO}_n(k)$ consists of the matrices in O_n that have determinant 1. The vector space E is the natural module for GL_n and for O_n and SO_n. We denote the maximal torus of SO_n that consists of the diagonal matrices by T_1. Then T_1 consists of the diagonal matrices t with $t_i t_{i'} = 1$ for all $i \in \{1, \ldots, m\}$ and with $t_{m+1} = 1$ in case n is odd. The character group of T_1 can be identified with $\mathbb{Z}^m$. The root system of SO_n with respect to T_1 is of type B_m if n is odd and of type D_m if n is even. We choose the set of roots of T_1 in the Lie algebra of the Borel subgroup of upper triangular matrices in SO_{2n} as the system of positive roots. A weight $\lambda \in \mathbb{Z}^m$ is dominant if and only if $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{m-1} \geq \lambda_m \geq 0$ in case n is odd and it is dominant if and only if $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{m-1} \geq |\lambda_m|$ in case n is even. We define the ε_i as in Sect. 1.1. We note that the group of weights of SO_n is a proper subgroup of the weight lattice of the root system, since SO_n is not simply connected. For λ a dominant weight of SO_n we denote the corresponding induced module by $\nabla_1(\lambda)$. The Schur algebra, Schur functor and inverse Schur functor are as defined in Sects. 1 and 2. The definitions and results for the Schur algebra given there are of course also valid for odd n. Let r be an integer ≥ 0. We define the *orthogonal Schur algebra* $S_1(n, r)$ to be the enveloping algebra of O_n in $\mathrm{End}_k(E^{\otimes r})$.

Let $B_r = B_r(n)$ be the Brauer algebra. There is a natural homomorphism $B_r \to \mathrm{End}_{\mathrm{O}_n}(E^{\otimes r})$. As in the symplectic case, one shows using classical invariant theory that this homomorphism is surjective and that it is injective if $n \geq 2r$. This is completely analogous to the symplectic case, see [39, Sect. 3]. In [8] a bideterminant basis is given for $k[\mathrm{O}_n]$. Cliff has informed us that his arguments also show that $k[\mathrm{Mat}_n]$ modulo the relations (60) in [22] is spanned by O_n-standard bideterminants multiplied by a power of the coefficient of dilation. From this one deduces that these relations generate the vanishing ideal of the orthogonal monoid. Now it follows that $\mathrm{End}_{B_r}(E^{\otimes r}) = S_1(n, r)$ by [39, Rem. 3.3]. See [23] for another approach to the double centralizer theorem for the orthogonal group.

By [7, Prop. 3.3(iii)] every GL_n-module with a good GL_n-filtration also has a good SO_n-filtration. From this one easily deduces that for every partition λ of length at most m, restriction of functions defines an epimorphism $\nabla(\lambda) \to \nabla_1(\lambda)$ (Note that in case n is even, SO_n also has other dominant weights). It also follows that a tilting module for GL_n is a tilting module for SO_n, by restriction. Now let $S_1^0(n, r)$ be the enveloping algebra of SO_n in $\mathrm{End}_k(E^{\otimes r})$. Clearly $S_1^0(n, r)^*$ is the coefficient space of the SO_n-module $E^{\otimes r}$. Recall the definition of $\Lambda_0^+(m, r)$ from Sect. 1.1. We now consider the cases n is even and n is odd separately.

First assume that n is even and $> 2r$. Then $m > r$ and it is not hard to check that the set $\Lambda_0^+(m, r)$ is saturated. The arguments in [20, Sect. 8] now show that $S_1^0(n, r)^* = O_{\Lambda_0^+(m,r)}(k[\mathrm{SO}_n])$ which means that $S_1^0(n, r)$ is the generalized Schur algebra associated to SO_n and $\Lambda_0^+(m, r)$. We will now show that $S_1^0(n, r) = S_1(n, r)$. The dimension of $S_1^0(n, r)$ is independent of the field by [13, (2.2c)]. By [39, Prop. 1(i)] the vector space $S_1(n, r)$ is isomorphic to the dual of the vector space $A_{\mathrm{GO}}(n, r, k)$ in [8, Thm. 8.1], the dimension of which is independent of the field. So we may now assume that $\mathrm{char}\, k = 0$. Then the algebras $S_1^0(n, r)$ and $S_1(n, r)$ are semisimple, so it suffices to check that their centralizers coincide. By the first fundamental theorem of invariant theory for SO_n, [41, Sect. II.9] or [11, Thm. 5.6(ii)], we have that $k[\oplus^{2r} E]^{\mathrm{SO}_n} = k[\oplus^{2r} E]^{\mathrm{O}_n}$, since $n > 2r$. Taking the multilinear invariants on both sides we obtain $\mathrm{End}_{\mathrm{SO}_n}(E^{\otimes r}) \cong (E^{\otimes 2r})^{*,\mathrm{SO}_n} = (E^{\otimes 2r})^{*,\mathrm{O}_n} \cong \mathrm{End}_{\mathrm{O}_n}(E^{\otimes r})$. The two isomorphisms here were observed by Brauer in [4].

Now we treat the case n is odd. In this case the set $\Lambda_0^+(m, r)$ is clearly not saturated, since $(r-1)\varepsilon_1 \le r\varepsilon_1 \in \Lambda_0^+(m, r)$ and $(r-1)\varepsilon_1 \notin \Lambda_0^+(m, r)$. So we will proceed differently. First note that $S_1^0(n, r) = S_1(n, r)$, since $-\mathrm{id} \in \mathrm{O}_n \setminus \mathrm{SO}_n$.

Let S be the enveloping algebra of O_n in $\mathrm{End}_k(E^{\otimes r} \oplus E^{\otimes (r-1)})$. Recall that the coordinate ring $k[\mathrm{Mat}_n]$ of Mat_n is graded and therefore filtered. Each filtration subspace is a subcoalgebra of $k[\mathrm{Mat}_n]$. The coordinate ring $k[\mathrm{SO}_n]$ of SO_n inherits a filtration from $k[\mathrm{Mat}_n]$, we denote the rth filtration subspace by $k[\mathrm{O}_n]^{\le r}$. This is a subcoalgebra of $k[\mathrm{SO}_n]$. By [39, Prop. 1(ii)] we have that $S^* \cong k[\mathrm{O}_n]^{\le r}$ (the summands $E^{\otimes s}$, $s < r-1$, occurring in [39] can be omitted). The defining ideal of O_n is homogeneous for the $\mathbb{Z}_2$-grading of $k[\mathrm{Mat}_n]$ (the two graded subspaces are the sums of the $\mathbb{Z}$-graded subspaces of even and odd degree, respectively), so $k[\mathrm{O}_n]^{\le r}$ has a direct sum decomposition into two sub coalgebra summands, and therefore S has a direct sum decomposition in two ideal summands: $S = S(0) \oplus S(1)$. We have $S_1(n, r) = S(0)$ if r is even and $S_1(n, r) = S(1)$ if r is odd. The bideterminant basis of $k[\mathrm{O}_n]$ in [8, Cor 6.2] gives a basis for the subspace $k[\mathrm{O}_n]^{\le r}$ which is labelled by pairs of O_n-standard tableaux of some shape λ with $|\lambda| \le r$. In particular $k[\mathrm{O}_n]^{\le r}$, and therefore S has dimension independent of the field.

Now let S^0 be the enveloping algebra of SO_n in $\mathrm{End}_k(E^{\otimes r} \oplus E^{\otimes (r-1)})$ and let π be the set of dominant weights $\{\lambda \in \Lambda^+(m) \mid |\lambda| \le r\}$. The set π is clearly saturated and one checks using the arguments in [20, Sect. 8] that S^0 is the generalized Schur algebra associated to SO_n and π.

Now assume furthermore that $n > 2r$. Then we deduce as in the case n is even that $S^0 = S$. This also means that the restriction of functions $k[\mathrm{O}_n]^{\le r} \to k[\mathrm{SO}_n]^{\le r}$ is an isomorphism. So S is the generalized Schur algebra associated to SO_n and π, and one easily checks that the direct sum decomposition $S = S(0) \oplus S(1)$ corresponds to the partition of π into the sets $\Lambda_0^+(m, r)$ and $\pi \setminus \Lambda_0^+(m, r)$. So $S_1(n, r) = O_{\Lambda_0^+(m,r)}(k[\mathrm{SO}_n])^*$ is a direct ideal summand of the generalized Schur algebra S and therefore a quasihereditary algebra; $\Lambda_0^+(m, r)$ is the set of labels of its irreducibles. The point here is that in a quasihereditary algebra one can always make labels of irreducibles that belong to different blocks incomparable without affecting anything. This shows that a block, and therefore any direct ideal summand of a quasihereditary algebra is again quasihereditary. From the arguments in [13, 2.2] it is now also clear that $S_1(n, r)$ has dimension $\sum_{\lambda\in\Lambda_0^+(m,r)} \dim(\nabla_1(\lambda))^2$, which is independent of the field k.

In the remainder of this section we assume that $n > 2r$.

We define the orthogonal Schur functor $f_1 : \mathrm{mod}(S_1(n, r)) \to \mathrm{mod}(B_r)$ and the *inverse orthogonal Schur functor* $g_1 : \mathrm{mod}(B_r) \to \mathrm{mod}(S_1(n, r))$ in precisely the same way as in the symplectic case.

The proofs of the orthogonal versions of Lemmas 2.1–2.3 are completely analogous. Once the orthogonal version of (*) in Lemma 2.1 is proved for all r with $2(r + 1) < n$, the rest of the proof is the same as in the symplectic case. For the weight space argument in Lemma 2.2(ii) one can use the orthogonal standard tableaux from [31]. Now we obtain the orthogonal version of Theorem 2.1.

Theorem 4.1 *The following holds.*

(i) *For $\lambda \in \Lambda_0^+(m, r)$ we have*

$$\begin{aligned} f_1(\nabla_1(\lambda)) &\cong \mathcal{S}(\lambda),\\ f_1(S^\lambda E) &\cong \mathcal{M}(\lambda), \quad \textit{and}\\ f_1\Big(\bigwedge\nolimits^\lambda E\Big) &\cong \widetilde{\mathcal{M}}(\lambda) \quad \textit{if}\ \mathrm{char}\, k = 0 \ \textit{or} > |\lambda|. \end{aligned}$$

(ii) *Let M be an $S(n, t)$-module. If M is a direct sum of direct summands of $E^{\otimes t}$ or if M is injective, then*

$$f_1(M) \cong \mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} f(M).$$

For $\lambda \in \Lambda_0^+(m, r)$ we denote the indecomposable tilting module for SO_n of highest weight λ by $T_1(\lambda)$ and for λ p-regular we denote the projective cover of the irreducible B_r-module $\mathcal{D}(\lambda)$ by $\mathcal{P}(\lambda)$. The proof of the orthogonal version of Proposition 2.1 and its corollary are completely analogous.

Proposition 4.1 *Let $\lambda \in \Lambda_0^+(m, r)$. Then $T_1(\lambda)$ is a direct summand of the SO_n-module $E^{\otimes r}$ if and only if λ is p-regular and $\lambda \neq \emptyset$ in case r is even ≥ 2 and $\delta = 0$. Now assume that λ satisfies these conditions. Then*

(i) $f_1(T_1(\lambda)) = \mathcal{P}(\lambda)$.
(ii) *The multiplicity of $T_1(\lambda)$ in $E^{\otimes r}$ is* $\dim \mathcal{D}(\lambda)$.
(iii) *The decomposition number $[\mathcal{S}(\mu) : \mathcal{D}(\lambda)]$ is equal to the Δ-filtration multiplicity $(T_1(\lambda) : \Delta_1(\mu))$ and to the ∇-filtration multiplicity $(T_1(\lambda) : \nabla_1(\mu))$.*

For a fixed integer m' and a partition λ with $l(\lambda) \leq m$ and $l(\lambda') = \lambda_1 \leq m'$ we define $\lambda^\dagger$ as in Sect. 2. In the corollary below we apply the previous proposition in the case that $n = 2m$ is even.

Corollary *Let $\lambda, \mu \in \Lambda_0^+(r, r)$ with λ p-regular. Assume that $\lambda_1, \mu_1 \leq m'$. Then we have the equality of decomposition numbers*

$$\big[\mathcal{S}(\mu) : \mathcal{D}(\lambda)\big] = \big[\nabla'_1(\lambda^\dagger) : L'_1(\mu^\dagger)\big],$$

where ∇'_1 and L'_1 denote induced and irreducible modules for $\mathrm{SO}_{2m'}$.

The orthogonal versions of Remarks 2.1, 1, 2 and 4 and Lemma 2.4 can be proved in precisely the same way. We have the orthogonal version of (6)

$$f_1(A_1(n,r)) = \mathrm{Hom}_{\mathrm{SO}_n}(E^{\otimes r}, S_1(n,r)^*) \cong \mathrm{Hom}_{\mathrm{SO}_n}(S_1(n,r), E^{\otimes r}) \cong E^{\otimes r} \text{ and}$$
$$g_1(E^{\otimes r}) = E^{\otimes r} \otimes_{B_r} E^{\otimes r} \cong \mathrm{End}_{B_r}(E^{\otimes r})^* = S_1(n,r)^* \cong A_1(n,r). \tag{9}$$

Now we obtain the orthogonal version of Proposition 2.2.

Proposition 4.2

(i) *If $n = 0$ in k and $t = 0$, assume $r \geq 4$. Then we have*

$$g_1(\mathrm{Ind}_{k\mathrm{Sym}_t}^{B_r} V) \cong g(V)$$

as SO_n-modules, for every $k\mathrm{Sym}_t$-module V.

(ii) *Let $\lambda \in \Lambda_0^+(m,r)$. If $\lambda = \emptyset$ and $m = 0$ in k, then assume $r \geq 4$. Then $g_1(\mathcal{M}(\lambda)) \cong S^\lambda E$ and $g_1(\widetilde{\mathcal{M}}(\lambda)) \cong \bigwedge^\lambda E$.*

(iii) *Let $\lambda \in \Lambda_0^+(m,r)$. The SO_n-module $S^\lambda E$ has a unique indecomposable summand $J(\lambda)$ in which $\nabla_1(\lambda)$ has filtration multiplicity > 0 and this multiplicity is equal to 1. Every summand of $\mathcal{M}(\lambda)$ has a Specht filtration and $f_1(J(\lambda)) \cong \mathcal{Y}(\lambda)$.*

We finally note that the orthogonal versions of Remarks 2.2 are also valid. Only Remark 2.2.3 needs some modifications in the case that n is odd.

5 Blocks

5.1 The blocks of the Brauer algebra and the symplectic and orthogonal Schur algebras in characteristic p

In this section we assume that the field k is of positive characteristic p. Furthermore, n is an integer ≥ 2 and we put $m = \lfloor n/2 \rfloor$. Let δ be an integer. Recall from Sect. 3 (the paragraph before Proposition 3.1) that the block relation is defined on a labelling set for the irreducibles. Since cell modules of a cellular algebra always belong to one block, we can extend the block relation of $B_r(\delta)$ to an equivalence relation on all of $\Lambda_0^+(r,r)$ (not just the p-regular partitions) as follows: λ and μ are in the same block if and only if $\mathcal{S}(\lambda)$ and $\mathcal{S}(\mu)$ belong to the same block. Note that if $p > r$, we are only extending the block relation if r is even ≥ 2 and δ is zero in k. In this case we add the empty partition. Let λ be a partition of t, $t \leq r$ with $r - t = 2s$ even. We have $k_{\mathrm{sg}} \otimes S(\lambda) \cong S(\lambda')^*$. Since every simple $k\mathrm{Sym}_t$-module is self-dual, we have that V and V^* have the same composition factors (with multiplicities) for every finite dimensional $k\mathrm{Sym}_t$-module V. So $k_{\mathrm{sg}} \otimes S(\lambda)$ and $S(\lambda')$ are in the same $k\mathrm{Sym}_t$-block. Since the functor $Z_s \otimes_{k\mathrm{Sym}_t} -$ is exact we get that $\widetilde{\mathcal{S}}(\lambda)$ and $\mathcal{S}(\lambda')$ are in the same $B_r(\delta)$-block.

To prove our next result, we need the following basic fact about quasihereditary algebras. For lack of reference we include a proof. For the general theory of quasihereditary algebras we refer to the appendix of [19].

Lemma 5.1 *Let S be a finite dimensional quasihereditary algebra with partially ordered labelling set $(X, \leq)$ for the irreducibles. Let S' be the Ringel dual of S with reversed partial order $\leq' = \leq^{\mathrm{op}}$ on the labelling set X and let $\lambda, \mu \in X$. Then λ and μ are in the same S-block if and only if they are in the same S'-block.*

Proof We have $S' = \mathrm{End}_S(T)^{\mathrm{op}}$ for a full tilting module T of S. Denote the irreducible, indecomposable projective and indecomposable tilting module with label λ by $L(\lambda)$, $P(\lambda)$ and $T(\lambda)$, respectively. The analogues for S' are "primed". Recall that we have the canonical functor $F = \mathrm{Hom}_S(T, -) : \mathrm{mod}(S) \to \mathrm{mod}(S')$. Since $(S')'$ is Morita-equivalent to S, it suffices to show that λ and μ are in the same S-block if they are in the same S'-block. The block relation of S' is generated by the relation $[P'(\lambda) : L'(\mu)] \neq 0$, so it suffices to show that $[P'(\lambda) : L'(\mu)] \neq 0$ implies that λ and μ are in the same S-block. So assume the former. We have $[P'(\lambda) : L'(\mu)] = \dim \mathrm{Hom}_{S'}(P'(\mu), P'(\lambda))$. Since $F(T(\nu)) = P'(\nu)$ for every $\nu \in X$, we have that $\mathrm{Hom}_S(T(\mu), T(\lambda)) \neq 0$, by the isomorphism [19, Prop. A4.8(i)]. This clearly implies that λ and μ are in the same S-block. □

Theorem 5.1 *Let* $\lambda, \mu \in \Lambda_0^+(r, r)$.

(i) *Assume n is even and* $m \geq r$. *Then* λ *and* μ *are in the same* $S_0(n, r)$*-block if and only if* λ' *and* μ' *are in the same* $B_r(-n)$*-block.*
(ii) *Assume* $p \neq 2$ *and* $n > 2r$. *Then* λ *and* μ *are in the same* $S_1(n, r)$*-block if and only if they are in the same* $B_r(n)$*-block.*

Proof (i). First assume that λ' and μ' are in the same $B_r(-n)$-block. As we have seen, this means that $\widetilde{S}(\lambda)$ and $\widetilde{S}(\mu)$ are in the same $B_r(-n)$-block. Let T be the full tilting module $\bigoplus_{\nu \in \Lambda_0^+(r,r)} \bigwedge^\nu E$ and let $S' = \mathrm{End}_{S_0(n,r)}(T)$ be the Ringel dual of $S_0(n, r)$. Let $F : \mathrm{mod}(S_0(n, r)) \to \mathrm{mod}(S')$ be as above. Denote the projection of T onto $E^{\otimes r}$ by e. Then we have $eS'e = B_r(-n)$ and $f_0(M) = eF(M)$ for every $S_0(n, r)$-module M. By Theorem 2.1(i) we have $e\Delta'(\nu) = \widetilde{S}(\nu)$ for all $\nu \in X$. Here we have used that $F(\nabla_0(\nu)) = \Delta'(\nu)$, the standard module with label ν of S'. By assumption there exists an indecomposable summand (block) A of $B_r(-n)$ such that A is nonzero on $\widetilde{S}(\lambda)$ and $\widetilde{S}(\mu)$. Let R be the indecomposable summand of S' such that $A \subseteq eRe$. Then R is nonzero on $\Delta'(\lambda)$ and $\Delta'(\mu)$. It follows that λ and μ are in the same S'-block and therefore also in the same $S_0(n, r)$-block, by Lemma 5.1.

Now we will show that λ' and μ' are in the same $B_r(-n)$-block if λ and μ are in the same $S_0(n, r)$-block. Since the relation $[\Delta_0(\mu) : L_0(\lambda)] \neq 0$ generates the block relation of $S_0(n, r)$, we may assume that $[\Delta_0(\mu) : L_0(\lambda)] \neq 0$. Then $(I_0(\lambda) : \nabla_0(\mu)) = [\Delta_0(\mu) : L_0(\lambda)] \neq 0$; see, e.g. [19, Prop. A2.2]. Here $I_0(\lambda) \subseteq A_0(n, r)$ denotes the $S_0(n, r)$-injective hull of $\nabla_0(\lambda)$. Applying the symplectic Schur functor we obtain that $\widetilde{S}(\mu)$ occurs in a twisted Specht filtration of $f_0(I_0(\lambda))$. By (6) we have that $f_0(I_0(\lambda))$ is indecomposable; see also Remark 2.2. Since, clearly, $\widetilde{S}(\lambda)$ is a submodule of $f_0(I_0(\lambda))$, we get that $\widetilde{S}(\lambda)$ and $\widetilde{S}(\mu)$ are in the same $B_r(-n)$-block. As we have seen, this means that λ' and μ' are in the same $B_r(-n)$-block.

The proof of (ii) is completely analogous. Here we use Theorem 4.1(i) and (9) instead of Theorem 2.1(i) and (6). □

In the corollaries below the star action of $W(D_r)$ is defined as in Sect. 3. The definitions of $\hat{\rho}$ and the star action given there make sense for any integer δ.

Corollary 1 *Let* δ *be an integer. Assume that* $p \neq 2$ *and that* $|\delta| + 2r < p/2$. *Let* $\lambda, \mu \in \Lambda_0^+(r, r)$. *Then* λ *and* μ *are in the same block of* $B_r(\delta)$ *if and only if* λ' *and* μ' *are conjugate under the star action of* $W(D_r)$.

Proof Choose and integer u such that $\delta - up = -2m$, where $m \geq r$. Now apply Theorem 5.1(i) and Theorem 3.2. □

Corollary 2 *Let δ be the unique integer with $|\delta| < p/2$ and $n-\delta \in p\mathbb{Z}$. Assume that $p \neq 2$, that $n > 2r$ and that $|\delta| + 2r < p/2$. Let $\lambda, \mu \in \Lambda_0^+(r,r)$. Then λ and μ are in the same block of $S_1(n,r)$ if and only if λ' and μ' are conjugate under the star action of $W(D_r)$.*

Proof This follows immediately from Theorem 5.1(ii) and the preceding corollary. □

Remarks 5.1 1. Assume $p \neq 2$. Let δ be any integer and let r be an integer ≥ 0. Let $W_p(D_r)$ be the affine Weyl group of type D_r. One can define the star action of $W_p(D_r)$ in the same way as for $W(D_r) \subseteq W(C_r)$. Cox, De Visscher and Martin [10, Cor. 6.3] obtained the following linkage principle. Let $\lambda, \mu \in \Lambda_0^+(r,r)$. Then λ' and μ' are conjugate under the star action of $W_p(D_r)$ if λ and μ are in the same $B_r(\delta)$-block. From Theorem 5.1(i) we now deduce that for $n = 2m \geq 2r$, λ and μ are conjugate under the star action of $W_p(D_r)$ if they are in the same $S_0(n,r)$-block; here δ is the unique integer with $|\delta| < p/2$ and $-n-\delta \in p\mathbb{Z}$. Similarly, we deduce that for $n > 2r$, λ' and μ' are conjugate under the star action of $W_p(D_r)$ if λ and μ are in the same $S_1(n,r)$-block; here δ is the unique integer with $|\delta| < p/2$ and $n - \delta \in p\mathbb{Z}$.

From the linkage principles for the symplectic and orthogonal group we could deduce similar, but weaker linkage principles.

2. Assume that $p > r$. Then the Schur algebra $S(n,r)$ is semisimple. Now assume further that $n = 2m$ is even. Recall that $\bigoplus_{\lambda \in \Lambda_0^+(m,r)} \bigwedge^\lambda E$ is a full tilting module for $S_0(n,r)$. The natural epimorphisms $E^{\otimes t} \to \bigwedge^\lambda E$ are split, since they are $S(n,r)$-epimorphisms. From Lemma 1.1 we now deduce that $E^{\otimes r}$ is a full tilting module if $n \neq 0$ in k and that, in general, $E^{\otimes r} \oplus k$ is a full tilting module. In particular, if $n = 2m$, $p > r$ and $n \neq 0$ in k, then the Brauer algebra $B_r(-n)$ is the Ringel dual of the symplectic Schur algebra $S_0(n,r)$. Similar remarks apply to $B_r(n)$ and orthogonal Schur algebra $S_1(n,r)$.

5.2 Generalities on reduction mod p

We shall need that, for a fixed integer δ, the blocks of the Brauer algebra over a field of characteristic zero "agree" with the blocks over a field of large prime characteristic. In this section we recall the general reduction argument. The notation used here is completely independent from that in the rest of the paper. Let R be a Dedekind domain with field of fractions K. We fix a finite dimensional K-algebra A and an order Λ in A. Thus Λ is an R-subalgebra of A which is finitely generated as an R-module and the K-span of Λ is A. We assume that A is split, in the sense that $\mathrm{End}_A(V) = K$ for every irreducible A-module V. We will show that the separation into blocks of the irreducible modules over K agrees with that of $\Lambda/M\Lambda$, for all but finitely many maximal ideals M of R.

By a lattice we mean a Λ-module that is finitely generated and torsion free as an R-module. If V is an A-module of finite K-dimension we shall say that a Λ-submodule L of V is a (full) lattice in V if L is finitely generated over R and the K-span of L is V.

We write $\mathrm{Max}(R)$ for the set of maximal ideals of R. For $M \in \mathrm{Max}(R)$ and $F = R/M$ we have the finite dimensional F-algebra $\Lambda_F = F \otimes_R \Lambda$. If L is a lattice over Λ then we obtain a finite dimensional Λ_F-module $L_F = F \otimes_R L$ by base change.

We will use that an exact sequence $0 \to L_1 \to L_2 \to L_3 \to 0$ of R-modules splits if L_3 is finitely generated and torsion free. This follows immediately from the fact that a finitely generated torsion free module over a Dedekind ring is projective.

For a finite dimensional algebra S we write $\mathrm{Grot}(S)$ for the Grothendieck group of finitely generated left S-modules. We write $[V]$ for the class in $\mathrm{Grot}(S)$ of a finitely generated S-module V. Recall that if V is a finite dimensional A-module and L is a lattice in V

then the class $[L_F]$ is independent of the choice of the lattice L. (See, for example, the argument of Sect. 15.1, Théorème 32 of [37].) We have the decomposition homomorphism $\mathrm{Grot}(A) \to \mathrm{Grot}(\Lambda_F)$, taking $[V]$ to $[L_F]$.

We now label the set of maximal ideals M_t, $t \in T$ (with $M_s \neq M_t$ for $s \neq t$). We set $F_t = R/M_t$ and $A_t = \Lambda_{F_t}$, for $t \in T$. We fix a complete set of pairwise irreducible A-modules $V_1, \ldots, V_n$ and choose corresponding lattices $L_1, \ldots, L_n$ in these modules. Let $d_i = \dim V_i$, $1 \leq i \leq n$.

We note the following.

(1) *For all but finitely many $t \in T$, the algebra Λ_{F_t} is split and $L_{1,F_t}, \ldots, L_{n,F_t}$ is a complete set of irreducible pairwise non-isomorphic A_t-modules.*

First suppose A is semisimple. Let $e(i)$ be the central idempotent that acts as the identity on V_i and as 0 on V_j, for $j \neq i$. We have the orthogonal decomposition $1 = e(1) + \cdots + e(n)$ of $1 \in A$ as a sum of centrally primitive idempotents.

For $1 \leq k \leq n$, the algebra $e(k)A$ is a $d_k \times d_k$ matrix algebra. We choose a total matrix basis $e(k)_{ij}$, $1 \leq i, j \leq d_k$, of $e(k)A$, for $1 \leq k \leq m$. For some $0 \neq g \in R$, we have $ge(k)_{ij} \in \Lambda$ for all k, i, j. Now g is contained in only finitely many maximal ideals. If M_t is a maximal ideal not containing g and $\overline{g} = g + M$ then defining elements $f(k) = \overline{g}^{-1}(1 \otimes ge(k))$ and $f(k)_{ij} = \overline{g}^{-1}(1 \otimes ge(k)_{ij})$, of A_t, we have an orthogonal idempotent decomposition $1 = f(1) + \cdots + f(n)$ in A_t and a total matrix basis $f(k)_{ij}$, $1 \leq i, j \leq n_k$, of $A_t f(k)$, for $1 \leq k \leq n$. In particular $A_t f(k)$ is a matrix algebra, $f(k)$ is centrally primitive, $1 \leq k \leq n$, and A_t has n pairwise non-isomorphic simple modules. Now $f(k)$ acts as the identity on L_{k,F_t} and L_{k,F_t} has F_t-dimension d_k. Hence $L_{k,F}$ is simple and absolutely irreducible as a $A_t f(k)$-module and hence as a A_t-module. Thus $L_{1,F}, \ldots, L_{n,F}$ is a complete set of pairwise non-isomorphic simple A_t-modules and A_t is split, for almost all values of t.

We now consider the general case. Let J be the Jacobson radical of A and put $I = J \bigcap \Lambda$. Then I acts annihilates each V_i and hence each L_i. Let $t \in T$. We identify $I_{F_t} = F_t \otimes_R I$ with an ideal of A_t and Λ_{F_t}/I_{F_t} with $(\Lambda/I)_{F_t}$. Then I_{F_t} is a nilpotent ideal of A_t and I_{F_t} acts as 0 on each L_{i,F_t}. By the case already considered, for all but finitely many values of t, the modules $L_{1,F_t}, \ldots, L_{n,F_t}$ form a compete set of pairwise non-isomorphic simple Λ_{F_t}/I_{F_t}-modules all of which are absolutely irreducible. Hence the modules $L_{1,F_t}, \ldots, L_{n,F_t}$ form a compete set of pairwise non-isomorphic simple A_t-modules and A_t is split.

Let T^0 be the set of those $t \in T$ such that Λ_{F_t} is split and $L_{1,F_t}, \ldots, L_{n,F_t}$ form a complete set of pairwise non-isomorphic Λ_{F_t}-modules. For $t \in T^0$ and $1 \leq i \leq n$, we set $V_{it} = L_{i,F_t}$. Thus $V_{1t}, \ldots, V_{nt}$ is a complete set of pairwise non-isomorphic irreducible A_t-modules, for $t \in T^0$. For $1 \leq i, j \leq n$ we have the Cartan invariant c_{ij} of A, i.e. the composition multiplicity of V_j in the projective cover of V_i. For $t \in T^0$, we have the corresponding Cartan invariant c_{ij}^t of Λ_{F_t}.

(2) *For all but finitely many values of $t \in T^0$ we have $c_{ij} = c_{ij}^t$ for all $1 \leq i, j \leq n$ and in particular the modules V_k and V_l belong to the same block if and only if the modules V_{kt} and V_{lt} belong to the same block, for $1 \leq k, l \leq n$.*

For $1 \leq i \leq n$ we let P_i be the projective cover of V_i. For $t \in T^0$ we denote by P_{it} the projective cover of V_{it}. We choose a decomposition of $1 \in A$ as an orthogonal sum of primitive idempotents, $1 = \sum_{i=1}^{n} \sum_{j=1}^{d_i} e_{ij}$ such that Ae_{ij} is isomorphic to P_i, for $1 \leq i \leq n$, $1 \leq j \leq d_i$. We choose $0 \neq h \in R$ such that $he_{ij} \in \Lambda$, for all i, j.

We define T^1 to be the set of $t \in T^0$ such that $h \notin M_t$. Thus the set T^1 is cofinite in T. We have the lattice $Y_{ij} = h\Lambda e_{ij}$ in Ae_{ij}, for all i, j, and the lattice $Y = \bigoplus_{ij} Y_{ij}$ in A.

For $t \in T^1$, the inclusion $Y \to \Lambda$ induces an isomorphism $Y_{F_t} \to \Lambda_{F_t}$. Thus each $Y_{ij,F_t} = F_t \otimes_R Y_{ij}$ is a non-zero projective A_{F_t}-module. Moreover, the number of summands in a Λ_{F_t}-module decomposition of the left regular module Λ_{F_t} as a direct sum of indecomposable projective modules is $\sum_{i=1}^n d_i^2$. Hence each Y_{ij,F_t} is indecomposable and projective.

We fix $1 \le i \le n$ and $1 \le j \le d_i$. Let W be the maximal submodule of Ae_{ij} and let $H = W \bigcap Y_{ij}$. Then Ae_{ij}/W is isomorphic to V_i. Hence Y_{ij}/H is isomorphic to a lattice in V_i and $(Y_{ij}/H)_{F_t}$ is isomorphic to V_{it}, by (1). Thus the projective indecomposable A_t-module Y_{ij,F_t} has V_{it} as a homomorphic image and hence Y_{ij,F_t} is a projective cover of V_{it}, i.e. we have $Y_{ij,F_t} \cong P_{it}$.

By (1) the decomposition map $d_t : \mathrm{Grot}(A) \to \mathrm{Grot}(\Lambda_{F_t})$ is an isomorphism, taking $[V_i]$ to $[V_{it}]$, $1 \le i \le n$. Moreover, d_t takes $[P_i]$ to $[Y_{ij,F_t}] = [P_{it}]$. Now we have $[P_i] = \sum_{j=1}^n c_{ij}[V_j]$ and applying d_t we obtain $[P_{it}] = \sum_{j=1}^n c_{ij}[V_{jt}]$, which shows that $c_{ij} = c_{ij}^t$, for all $1 \le i, j \le n$ and $t \in T^1$.

5.3 The blocks of the Brauer algebra in characteristic 0

In this final section we give a Lie theoretic proof of the block result of Cox, De Visscher and Martin. We assume that char $k = 0$. Furthermore δ is an arbitrary integer, r is an integer ≥ 0 and we define $\hat{\rho}$ and the star action of the Weyl group $W(D_r) \subseteq W(C_r)$ as in Sect. 3. If r is even ≥ 2 and $\delta = 0$, we extend the block relation of $B_r(\delta)$ to all of $\Lambda_0^+(r, r)$ as in Sect. 5.1. We will apply the general results of Sect. 5.2 to the case that $R = \mathbb{Z}$, $K = \mathbb{Q}$, $A = B_r(\delta)_{\mathbb{Q}}$ and $\Lambda = B_r(\delta)_{\mathbb{Z}}$.

Theorem 5.2 [10, Thm. 4.2] *Let $\lambda, \mu \in \Lambda_0^+(r, r)$. Then λ and μ are in the same $B_r(\delta)$-block if and only if λ' and μ' are conjugate under the star action of $W(D_r)$.*

Proof We have $B_r(\delta) = k \otimes_{\mathbb{Q}} B_r(\delta)_{\mathbb{Q}}$. Since $B_r(\delta)_{\mathbb{Q}}$ is split, i.e. all irreducibles are absolutely irreducible. From this one deduces easily that the block relation of $B_r(\delta)$ is the same as that of $B_r(\delta)_{\mathbb{Q}}$. The same holds for $B_r(\delta)_{\mathbb{F}_p}$ and $B_r(\delta)_{\overline{\mathbb{F}}_p}$, where p is any prime and $\overline{\mathbb{F}}_p$ is the algebraic closure of the prime field $\mathbb{F}_p$. By (1) and (2) in Sect. 5.2 we can choose a prime $p > 2(|\delta| + 2r)$ such that the irreducibles of $B_r(\delta)_{\mathbb{F}_p}$ are the reductions mod p of the irreducibles of $B_r(\delta)_{\mathbb{Q}}$ and such that both algebras have the same block relation. Note that $\mathcal{S}(\lambda)_{\mathbb{F}_p}$ is a reduction mod p of $\mathcal{S}(\lambda)_{\mathbb{Q}}$. One easily checks that $\mathcal{D}(\lambda)_{\mathbb{F}_p} := \mathrm{hd}\,\mathcal{S}(\lambda)_{\mathbb{F}_p}$ is the reduction mod p of $\mathcal{D}(\lambda)_{\mathbb{Q}}$, $\lambda \ne \emptyset$ in case r is even ≥ 2 and $\delta = 0$. Since, by Sect. 5.2(1), the decomposition homomorphism $\mathrm{Grot}(B_r(\delta)_{\mathbb{Q}}) \to \mathrm{Grot}(B_r(\delta)_{\mathbb{F}_p})$ is an isomorphism, we have that $\mathcal{D}(\lambda)_{\mathbb{F}_p}$ is a composition factor of $\mathcal{S}(\mu)_{\mathbb{F}_p}$ if and only if $\mathcal{D}(\lambda)_{\mathbb{Q}}$ is a composition factor of $\mathcal{S}(\mu)_{\mathbb{Q}}$. The result now follows immediately from Corollary 1 of Theorem 5.1. □

Remark Since the Brauer algebra is cellular over $\mathbb{Z}$, one can actually deduce equality of the block relations of $B_r(\delta)_{\mathbb{F}_p}$ and $B_r(\delta)_{\mathbb{Q}}$ whenever the irreducibles of $B_r(\delta)_{\mathbb{F}_p}$ are the reductions mod p of the irreducibles of $B_r(\delta)_{\mathbb{Q}}$.

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