CORRIGENDUM TO: ON THE FIRST RESTRICTED COHOMOLOGY OF A REDUCTIVE LIE ALGEBRA AND ITS BOREL SUBALGEBRAS

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SUMMARY. We correct a mistake in the proof of the results for the cohomology of the coordinate rings of the groups G and B.

All references below to results where the source isn't mentioned are to [T].

(1). The second reference to Lemma 1.2(2) in the proof of Lemma 1.3(1) should be to Lemma 1.2(1).

(2). The "local version of Lemma 1.3" mentioned after the proof of Lemma 1.3 and in the proof of Theorem 3.1 would need the ordinary Nakayama Lemma which only holds for finitely generated modules, so we cannot use this local version in the proof of Theorem 3.1. Instead we can use the following version of Lemma 1.3.

Lemma. *Let S be a Noetherian commutative ring, let R be a Noetherian subring of S which is a domain such that $R \cap \mathfrak{n} \in \mathrm{Maxspec}(R)$ for all $\mathfrak{n} \in \mathrm{Maxspec}(S)$, and let N be a finitely generated S-module which is flat over R.*

- (i) *Let M be an S-submodule of N with $(\mathfrak{m}N) \cap M \subseteq \mathfrak{m}M$ for all $\mathfrak{m} \in \mathrm{Maxspec}(R)$. Then N/M is a torsion-free R-module.*
- (ii) *Let M be an S-module, let $\varphi : M \to N$ be an S-linear map. Assume that for all $\mathfrak{m} \in \mathrm{Maxspec}(R)$ the canonical map $M \to M/\mathfrak{m}M$ maps $\mathrm{Ker}(\varphi)$ onto the kernel of the induced $R/\mathfrak{m}$-linear map $\overline{\varphi} : M/\mathfrak{m}M \to N/\mathfrak{m}N$. Then $N/\mathrm{Im}(\varphi)$ is a torsion-free R-module.*

Proof. (i). It is enough to show that $(N/M)_\mathfrak{n} = N_\mathfrak{n}/M_\mathfrak{n}$ is $R_\mathfrak{m}$-torsion-free, where $\mathfrak{m} = \mathfrak{n} \cap R$, for all $\mathfrak{n} \in \mathrm{Maxspec}(S)$. Now $N_\mathfrak{n}$ is flat over R by the associativity of the tensor product, and since it is an $R_\mathfrak{m}$-module, it is also flat over $R_\mathfrak{m}$. Furthermore, $(\mathfrak{m}_\mathfrak{m} N_\mathfrak{n}) \cap M_\mathfrak{n} \subseteq \mathfrak{m}_\mathfrak{m} M_\mathfrak{n}$. So we may assume that $(R, \mathfrak{m})$ is a local Noetherian domain, and $(S, \mathfrak{n})$ a local Noetherian R-algebra with $\mathfrak{m}S \subseteq \mathfrak{n}$.

The assumption on M means that the $R/\mathfrak{m}$-module map $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective. Since $\mathrm{Tor}_1^R(R/\mathfrak{m}, N) = 0$ we get that $\mathrm{Tor}_1^R(R/\mathfrak{m}, N/M) = 0$. By the local criterion for flatness (see e.g. [Eis, Thm 6.8]) this means that N/M is a flat R-module. So it is torsion-free by [Eis, Cor 6.3].
(ii). Follows from (i) as in the proof of Lemma 1.3. ☐

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Now we can correct the proof of Theorem 3.1 as follows. Let $R = k[G]^G$ be as in that proof. To show that $H^1(G_1, k[G])$ is R-torsion-free it suffices to show that the cokernel of $\varphi = f \mapsto (x \mapsto x \cdot f) : k[G] \to \mathrm{Hom}_k(\mathfrak{g}, k[G])$ is R-torsion-free. This follows by applying the second assertion of the above lemma with $S = k[G]^{\mathfrak{g}}$. The point is that $k[G]^{\mathfrak{g}}$ contains (R and) the p-th powers, so $\mathrm{Hom}_k(\mathfrak{g}, k[G])$ is a finitely generated $k[G]^{\mathfrak{g}}$-module. The proofs of Theorem 3.2 and 4.2(2) are corrected similarly. We take $(R, S) = \big(k[B]^B, k[B]^{\mathfrak{b}}\big)$ for Theorem 3.2 and $(R, S) = \big(k[G]^G, k[G]^{G_r}\big)$ and $(R, S) = \big(k[B]^B, k[B]^{B_r}\big)$ for the two statements in Theorem 4.2(2).

References

[Eis] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Math., vol. 150, Springer, New York, 1995.

[T] R. Tange, *On the first restricted cohomology of a reductive Lie algebra and its Borel subalgebras*, Ann. Inst. Fourier (Grenoble) **69** (2019), no. 3, 1295-1308.

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