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ON THE FIRST RESTRICTED COHOMOLOGY OF A REDUCTIVE LIE ALGEBRA AND ITS BOREL SUBALGEBRAS

by Rudolf TANGE (*)

ABSTRACT. — Let k be an algebraically closed field of characteristic $p > 0$ and let G be a connected reductive group over k. Let B be a Borel subgroup of G and let $\mathfrak{g}$ and $\mathfrak{b}$ be the Lie algebras of G and B. Denote the first Frobenius kernels of G and B by G_1 and B_1. Furthermore, denote the algebras of regular functions on G and $\mathfrak{g}$ by $k[G]$ and $k[\mathfrak{g}]$, and similarly for B and $\mathfrak{b}$. The group G acts on $k[G]$ via the conjugation action and on $k[\mathfrak{g}]$ via the adjoint action. Similarly, B acts on $k[B]$ via the conjugation action and on $k[\mathfrak{b}]$ via the adjoint action. We show that, under certain mild assumptions, the cohomology groups $H^1(G_1, k[\mathfrak{g}])$, $H^1(B_1, k[\mathfrak{b}])$, $H^1(G_1, k[G])$ and $H^1(B_1, k[B])$ are zero. We also extend all our results to the cohomology for the higher Frobenius kernels.

RÉSUMÉ. — Soit k un corps algébriquement clos de charactéristique $p > 0$ and soit G un groupe réductif connexe sur k. Soit B un sous-groupe de Borel de G et soit $\mathfrak{g}$ et $\mathfrak{b}$ les algèbres de Lie de G et B. Notons les premiers noyaux de Frobenius de G et B par G_1 et B_1. De plus, notons les algèbres des fonctions régulières sur G et $\mathfrak{g}$ par $k[G]$ et $k[\mathfrak{g}]$, et de même pour B et $\mathfrak{b}$. Le groupe G agit sur $k[G]$ par conjugaison et sur $k[\mathfrak{g}]$ par l'action adjointe. De même, B agit sur $k[B]$ par l'action de conjugaison et sur $k[\mathfrak{b}]$ par l'action adjointe. Nous montrons que, sous certaines hypothèses, les groupes de cohomologie $H^1(G_1, k[\mathfrak{g}])$, $H^1(B_1, k[\mathfrak{b}])$, $H^1(G_1, k[G])$ et $H^1(B_1, k[B])$ sont nuls. Nous étendons aussi nos résultats à la cohomologie pour les noyaux de Frobenius supérieurs.

Introduction

Let k be an algebraically closed field of characteristic $p > 0$, let G be a connected reductive group over k, and let $\mathfrak{g}$ be the Lie algebra of G. Recall that $\mathfrak{g}$ is a restricted Lie algebra: it has a p-th power map $x \mapsto x^{[p]} : \mathfrak{g} \to \mathfrak{g}$, see [3, I.3.1]. In the case of $G = \mathrm{GL}_n$ this is just the p-th matrix power. A

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$\mathfrak{g}$-module M is called *restricted* if $(x^{[p]})_M = (x_M)^p$ for all $x \in M$. Here x_M is the endomorphism of M representing x.

Recall that an element v of a $\mathfrak{g}$-module M is called a $\mathfrak{g}$-invariant if $x \cdot v = 0$ for all $x \in \mathfrak{g}$. We denote the space of $\mathfrak{g}$-invariants in M by $M^{\mathfrak{g}}$. The right derived functors of the left exact functor $M \mapsto M^{\mathfrak{g}}$ from the category of restricted $\mathfrak{g}$-modules to the category of vector spaces over k are denoted by $H^i(G_1, \cdot)$.

Let $k[\mathfrak{g}]$ be the algebra of polynomial functions on $\mathfrak{g}$. If one is interested in describing the algebra of invariants $(k[\mathfrak{g}]/I)^{\mathfrak{g}}$ for some $\mathfrak{g}$-stable ideal I of $k[\mathfrak{g}]$, then it is of interest to know if $H^1(G_1, k[\mathfrak{g}]) = 0$, because then we have an exact sequence

$$k[\mathfrak{g}]^{\mathfrak{g}} \to (k[\mathfrak{g}]/I)^{\mathfrak{g}} \to H^1(G_1, I) \to 0$$

by the long exact cohomology sequence. So, in this case, $(k[\mathfrak{g}]/I)^{\mathfrak{g}}$ is built up from the image of $k[\mathfrak{g}]^{\mathfrak{g}}$ in $k[\mathfrak{g}]/I$, and $H^1(G_1, I)$.

The paper is organised as follows. In Section 1 we state some results from the literature that we need to prove our main result. This includes a description of the algebra of invariants $k[\mathfrak{g}]^G$, the normality of the nilpotent cone $\mathcal{N}$, and some lemmas on graded modules over graded rings. In Section 2 we prove Theorems 2.1 and 2.2 which state that, under certain mild assumptions on p, $H^1(G_1, k[\mathfrak{g}])$ and $H^1(B_1, k[\mathfrak{b}])$ are zero. In Section 3 we prove Theorems 3.1 and 3.2 which state that, under certain mild assumptions on p, $H^1(G_1, k[G])$ and $H^1(B_1, k[B])$ are zero. In Section 4 we extend theses four theorems to the cohomology for the higher Frobenius kernels G_r and B_r, $r \geqslant 2$.

We briefly indicate some background to our results. For convenience we only discuss the G-module $k[\mathfrak{g}]$. As is well-known, under certain mild assumptions $k[\mathfrak{g}]$ has a good filtration, see [6] or [11, II.4.22]. So a natural first approach to prove that $H^1(G_1, k[\mathfrak{g}]) = 0$ would be that to show that $H^1(G_1, \nabla(\lambda)) = 0$ for all induced modules $\nabla(\lambda)$ that show up in a good filtration of $k[\mathfrak{g}]$. However, this isn't true: even for $p > h$, h the Coxeter number, one can easily deduce counterexamples from [1, Cor. 5.5] (or [11, II.12.15]).(1) It is also easy to see that we cannot have $H^i(G_1, k[\mathfrak{g}]) = 0$ for all $i > 0$: the trivial module k is direct summand of $k[\mathfrak{g}]$, and for $p > h$ we have $H^\bullet(G_1, k) \cong k[\mathcal{N}]$ where the degrees of $k[\mathcal{N}]$ are doubled, see [11, II.12.14].

The idea of our proof that $H^1(G_1, k[\mathfrak{g}]) = 0$ is as follows. Noting that $H^1(G_1, k[\mathfrak{g}])$ is a $k[\mathfrak{g}]^G$-module, we interpret a certain localisation of

(1) This approach *does* work when proving (the well-known fact) that $H^1(G, k[\mathfrak{g}]) = 0$.

$H^1(G_1, k[\mathfrak{g}])$ as the cohomology group of the coordinate ring of the generic fiber of the adjoint quotient map $\mathfrak{g} \to \mathfrak{g}//G$. It is easy to see that this cohomology group has to be zero, so we are left with showing that $H^1(G_1, k[\mathfrak{g}])$ is torsion-free over the invariants $k[\mathfrak{g}]^G$. To prove the latter we use Hochschild's characterisation of the first restricted cohomology group and a "Nakayama Lemma type result". The ideas of the proofs of the other main results are completely analogous.

1. Preliminaries

Throughout this paper k is an algebraically closed field of characteristic $p > 0$. For the basics of representations of algebraic groups we refer to [11].

1.1. Restricted representations and restricted cohomology

Let G be a linear algebraic group over k with Lie algebra $\mathfrak{g}$. Let G_1 be the first Frobenius kernel of G (see [11, Ch. I.9]). It is an infinitesimal group scheme with $\dim k[G_1] = p^{\dim(\mathfrak{g})}$. Its category of representations is equivalent to the category of restricted representations of $\mathfrak{g}$, see the introduction.

Let M be an G_1-module. By [7] (see also [11, I.9.19]) we have

$$(1.1) \quad H^1(G_1, M) = \{\text{restricted derivations} : \mathfrak{g} \to M\}/\{\text{inner derivations of } M\}.$$

Here a *derivation* from $\mathfrak{g}$ to M is a linear map $D : \mathfrak{g} \to M$ satisfying

$$D([x, y]) = x \cdot D(y) - y \cdot D(x)$$

for all $x, y \in \mathfrak{g}$. Such a derivation is called *restricted* if

$$D(x^{[p]}) = (x_M)^{p-1}(D(x))$$

for all $x \in \mathfrak{g}$, where x_M is the vector space endomorphism of M given by the action of x, and $-^{[p]}$ denotes the p-th power map of $\mathfrak{g}$. An *inner derivation* of M is a map $x \mapsto x \cdot u : \mathfrak{g} \to M$ for some $u \in M$. If M is restricted, then every inner derivation is restricted. Clearly $H^1(G_1, M)$ is an G-module with trivial $\mathfrak{g}$-action: If D is a derivation and $y \in \mathfrak{g}$, then $[y, D]$ is the inner derivation given by $D(y)$. Note also that $H^1(G_1, k[\mathfrak{g}])$ is a $k[\mathfrak{g}]^{\mathfrak{g}}$-module, since the restricted derivations $\mathfrak{g} \to k[\mathfrak{g}]$ form a $k[\mathfrak{g}]^{\mathfrak{g}}$-module and the map $f \mapsto (x \mapsto x \cdot f)$ from $k[\mathfrak{g}]$ to the restricted derivations $\mathfrak{g} \to k[\mathfrak{g}]$ is $k[\mathfrak{g}]^{\mathfrak{g}}$-linear.

1.2. Actions of restricted Lie algebras

Let $\mathfrak{g}$ be a restricted Lie algebra over k. Following [17] we define an action of $\mathfrak{g}$ on an affine variety X over k to be a homomorphism $\mathfrak{g} \to \mathrm{Der}_k(k[X])$ of restricted Lie algebras, where $\mathrm{Der}_k(k[X])$ is the (restricted) Lie algebra of k-linear derivations of $k[X]$. It is easy to see that this includes the case that X is a restricted $\mathfrak{g}$-module. If $\mathfrak{g}$ acts on X and $x \in X$, then we define $\mathfrak{g}_x$ to be the stabiliser in $\mathfrak{g}$ of the maximal ideal $\mathfrak{m}_x$ of $k[X]$ corresponding to x. In case X is a closed subvariety of a restricted $\mathfrak{g}$-module, then we have $\mathfrak{g}_x = \{y \in \mathfrak{g} \,|\, y \cdot x = 0\}$.

Lemma 1.1. — *Let $\mathfrak{g}$ be a restricted Lie algebra over k acting on a normal affine variety X over k. If $\max_{x\in X} \mathrm{codim}_{\mathfrak{g}}\, \mathfrak{g}_x = \dim X$, then $k[X]^{\mathfrak{g}} = k[X]^p$.*

Proof. — By [17, Thm. 5.2 (5)] we have $[k(X) : k(X)^{\mathfrak{g}}] = p^{\dim(X)}$. By [4, Cor. 3 to Thm. V.16.6.4] we have $[k(X) : k(X)^p] = p^{\dim(X)}$. So $k(X)^{\mathfrak{g}} = k(X)^p$, since we always have $\supseteq$. Clearly, $k(X)^p = \mathrm{Frac}(k[X]^p)$, $k(X)^{\mathfrak{g}} = \mathrm{Frac}(k[X]^{\mathfrak{g}})$ and $k[X]^{\mathfrak{g}}$ is integral over $k[X]^p$. Since X is normal variety, $k[X]^p \cong k[X]$ is a normal ring. It follows that $k[X]^{\mathfrak{g}} = k[X]^p$. □

1.3. Two lemmas on graded rings and modules

We recall a version of the graded Nakayama lemma which follows from [14, Ch. 13, Lem. 4, Ex. 3, Lem. 3].

Lemma 1.2 ([14, Ch. 13]). — *Let $R = \bigoplus_{i\geqslant 0} R^i$ be a positively graded ring with R^0 a field, let M be a positively graded R-module and let $(x_i)_{i\in I}$ be a family of homogeneous elements of M. Put $R^+ = \bigoplus_{i>0} R^i$.*

1. *If the images of the x_i in M/R^+M span the vector space M/R^+M over R^0, then the x_i generate M.*
2. *If M is projective and the images of the x_i in M/R^+M form an R^0-basis of M/R^+M, then $(x_i)_{i\in I}$ is an R-basis of M.*

Lemma 1.3. — *Let R be a positively graded ring with R^0 a field and let N be a positively graded R-module which is projective.*

1. *Let M be a submodule of N with $(R^+N) \cap M \subseteq R^+M$. Then M is free and a direct summand of N.*
2. *Let M be a positively graded R-module, let $\varphi : M \to N$ be a graded R-linear map and let $\overline{\varphi} : M/R^+M \to N/R^+N$ be the induced R^0-linear map. Assume the canonical map $M \to M/R^+M$ maps $\mathrm{Ker}(\varphi)$ onto $\mathrm{Ker}(\overline{\varphi})$. Then $\mathrm{Im}(\varphi)$ is free and a direct summand of N.*

Proof.

(1). — From the assumption it is immediate that the natural map $M/R^+M \to N/R^+N$ is injective. Now choose an R^0-basis $(\overline{x}_i)_{i\in I}$ of M/R^+M and extend it to a basis $(\overline{x}_i)_{i\in I\cup J}$ of N/R^+N. Let $(x_i)_{i\in I\cup J}$ be a homogeneous lift of this basis to N. Then this is a basis of N by Lemma 1.2 (2). Furthermore, $(x_i)_{i\in I}$ must span M by Lemma 1.2 (2). So M is (graded-) free and has the R-span of $(x_i)_{i\in J}$ as a direct complement.

(2). — By (1) it suffices to show that $(R^+N)\cap \mathrm{Im}(\varphi) \subseteq R^+\,\mathrm{Im}(\varphi)$. Let $x \in M$ and assume that $\varphi(x) \in R^+N$. Then $\overline{x} := x + R^+M \in \mathrm{Ker}(\overline{\varphi})$. By assumption there exists $x_1 \in \mathrm{Ker}(\varphi)$ such that $\overline{x} = \overline{x}_1$. Then $x - x_1 \in R^+M$ and $\varphi(x) = \varphi(x - x_1) \in R^+\,\mathrm{Im}(\varphi)$. □

There is also an obvious version of Lemma 1.3 (and of course of Lemma 1.2) for a local ring R: simply assume R local, omit the gradings everywhere and replace R^+ by the maximal ideal of R.

1.4. The standard hypotheses and consequences

In the remainder of this paper G is a connected reductive group over k and $\mathfrak{g}$ is its Lie algebra. Recall that $\mathfrak{g}$ is a restricted Lie algebra, see [3, I.3.1], we denote its p-th power map by $x \mapsto x^{[p]}$. The group G acts on $\mathfrak{g}$ and the nilpotent cone $\mathcal{N}$ via the adjoint action and on G via conjugation, and therefore it also acts on their algebras of regular functions: $k[\mathfrak{g}]$, $k[\mathcal{N}]$ and $k[G]$. Fix a maximal torus T of G and let $\mathfrak{t}$ be its Lie algebra. We fix an $\mathbb{F}_p$-structure on G for which T is defined and split over $\mathbb{F}_p$. Then $\mathfrak{g}$ has an $\mathbb{F}_p$-structure and $\mathfrak{t}$ is $\mathbb{F}_p$-defined. Denote the $\mathbb{F}_p$-defined regular functions on $\mathfrak{g}$ and $\mathfrak{t}$ by $\mathbb{F}_p[\mathfrak{g}]$ and $\mathbb{F}_p[\mathfrak{t}]$. We will need the following standard hypotheses, see [10, 6.3, 6.4] or [12, 2.6, 2.9]:

(H1) The derived group DG of G is simply connected,

(H2) p is good for G,

(H3) There exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$.

Put $G_x = \{g \in G \,|\, \mathrm{Ad}(g)(x) = x\}$ and $\mathfrak{g}_x = \{y \in \mathfrak{g} \,|\, [y, x] = 0\}$. Assuming (H1)–(H3) we have by [12, 2.9] that $\mathrm{Lie}(G_x) = \mathfrak{g}_x$ for all $x \in \mathfrak{g}$. See also [15, Sect. 2.1]. Put $n = \dim(T)$. We call $x \in \mathfrak{g}$ *regular* if $\dim G_x$ (or $\dim \mathfrak{g}_x$) equals n, the minimal value. Under assumptions (H1) and (H3) we have that $d\alpha \neq 0$ for all roots α, so restriction of functions defines an isomorphism $k[\mathfrak{g}]^G \xrightarrow{\sim} k[\mathfrak{h}]^W$, see [12, Prop. 7.12]. The set of regular semisimple elements in $\mathfrak{g}$ is the nonzero locus of the regular function f_{rs} on

$\mathfrak{g}$ which corresponds under the above isomorphism to the product of the differentials of the roots. Note that $f_{\rm rs} \in \mathbb{F}_p[\mathfrak{g}]$: $f_{\rm rs}$ is defined over $\mathbb{F}_p$.

Under assumptions (H1)–(H3) it follows from work of Demazure [5] that $k[\mathfrak{t}]^W$ is a polynomial algebra in n homogeneous elements defined over $\mathbb{F}_p$, see [10, 9.6 end of proof]. We denote the corresponding elements of $\mathbb{F}_p[\mathfrak{g}]$ by $s_1, \dots, s_n$. Assuming (H1)–(H3) the vanishing ideal of $\mathcal{N}$ in $k[\mathfrak{g}]$ is generated by the s_i, see [12, 7.14], and all regular orbit closures are normal, in particular $\mathcal{N}$ is normal, see [12, 8.5].

We call $g \in G$ *regular* if $G_g := \{h \in G \,|\, hgh^{-1} = g\}$ has dimension n, the minimal value. Restriction of functions defines an isomorphism $k[G]^G \xrightarrow{\sim} k[T]^W$, see [19, 6.4]. The set of regular semisimple elements in G is the nonzero locus of the regular function $f'_{\rm rs}$ on G which corresponds under the above isomorphism to $\prod_{\alpha \text{ a root}}(\alpha - 1)$. If G is semisimple, simply connected, then $k[G]^G$ is a polynomial algebra in the characters $\chi_1, \dots, \chi_n$ of the irreducible G-modules whose highest weights are the fundamental dominant weights. Furthermore, the schematic fibers of the adjoint quotient $G \to G/\!/G$ are reduced and normal and they are regular orbit closures. See [19] and [8, 4.24]. One can also deduce from (H1)–(H3) that $\mathrm{Lie}(G_g) = \mathfrak{g}_g := \{x \in \mathfrak{g} \,|\, \mathrm{Ad}(g)(x) = x\}$.

2. The cohomology groups $H^1(G_1, k[\mathfrak{g}])$ and $H^1(B_1, k[\mathfrak{b}])$

Throughout this section we assume that hypotheses (H1)–(H3) from Section 1.4 hold.

THEOREM 2.1. — $H^1(G_1, k[\mathfrak{g}]) = 0$.

Proof. — Let K be an algebraic closure of the field of fractions of $R := k[\mathfrak{g}]^G$. Since the action of $\mathfrak{g}$ on $k[\mathfrak{g}]$ is R-linear we have $H^1(G_1, k[\mathfrak{g}]) = H^1\big((G_1)_R, k[\mathfrak{g}]\big)$, where $-_R$ denotes base change from k to R, see [11, I.1.10]. So, by the Universal Coefficient Theorem [11, Prop. I.4.18], we have

$$K \otimes_R H^1(G_1, k[\mathfrak{g}]) = H^1\big((G_1)_K, K \otimes_R k[\mathfrak{g}]\big) = H^1\big((G_K)_1, K \otimes_R k[\mathfrak{g}]\big)\,.$$

For $i \in \{1, \dots, n\}$ denote the regular function on $\mathfrak{g}_K$ corresponding to $s_i \in k[\mathfrak{g}]$ by $\widetilde{s}_i$. Then $K \otimes_R k[\mathfrak{g}] = K[\mathfrak{g}_K]/(\widetilde{s}_1 - s_1, \dots, \widetilde{s}_n - s_n) = K[F]$, where $F \subseteq \mathfrak{g}_K$ is the fiber of the morphism

$$x \mapsto (\widetilde{s}_1(x), \dots, \widetilde{s}_n(x)) : \mathfrak{g}_K \to \mathbb{A}^n_K$$

over the point $(s_1, \dots, s_n) \in \mathbb{A}^n_K$. Let $f_{\rm rs} \in \mathbb{F}_p[\mathfrak{g}] \cap k[\mathfrak{g}]^G$ be the polynomial function from Section 1.4 with nonzero locus the set of regular semisimple elements in $\mathfrak{g}$, and let $\widetilde{f}_{\rm rs}$ be the corresponding polynomial function on

$\mathfrak{g}_K$. Then we have for all $x \in F$ that $\widetilde{f_{\mathrm{rs}}}(x) = f_{\mathrm{rs}} \neq 0$. So F consists of regular semisimple elements. By [20, Lem. 3.7, Thm. 3.14] this means that $F = G_K/S$ for some maximal torus S of G_K. In particular, $K[F]$ is an injective G_K-module. But then it is also injective as a $(G_K)_1$-module, see [11, Rem. I.4.12, Cor. I.5.13b)]. So $K \otimes_R H^1(G_1, k[\mathfrak{g}]) = H^i\big((G_K)_1, K[F]\big) = 0$ for all $i > 0$.

So it now suffices to show that $H^1(G_1, k[\mathfrak{g}])$ has no R-torsion. We are going to apply Lemma 1.3 (2) to the R-linear map

$$\varphi = f \mapsto (x \mapsto x \cdot f) : k[\mathfrak{g}] \to \mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])\,.$$

Here the grading of $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])$ is given by

$$\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])^i = \mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}]^i).$$

As explained in [12, 7.13, 7.14] the conditions of [16, Prop. 10.1] are satisfied under the assumptions (H1)–(H3), so $k[\mathfrak{g}]$ is a free R-module. So $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])$ is also a free R-module. We have $k[\mathfrak{g}]/R^+k[\mathfrak{g}] = k[\mathcal{N}]$, and

$$\overline{\varphi} = f \mapsto (x \mapsto x \cdot f) : k[\mathcal{N}] \to \mathrm{Hom}_k(\mathfrak{g}, k[\mathcal{N}]).$$

By [12, 6.3,6.4], we have $\min_{x\in\mathcal{N}} \dim \mathfrak{g}_x = n$ and $\dim \mathcal{N} = \dim \mathfrak{g} - n$. So from Lemma 1.1 it is clear that the restriction map $k[\mathfrak{g}] \to k[\mathcal{N}]$ maps the $\mathfrak{g}$-invariants of $k[\mathfrak{g}]$ onto those of $k[\mathcal{N}]$. But $k[\mathfrak{g}]^{\mathfrak{g}} = \mathrm{Ker}(\varphi)$ and $k[\mathcal{N}]^{\mathfrak{g}} = \mathrm{Ker}(\overline{\varphi})$. So, by Lemma 1.3 (2), $\mathrm{Im}(\varphi)$ is a direct R-module summand of $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])$. In particular, $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])/\,\mathrm{Im}(\varphi)$ is isomorphic to an R-submodule of $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])$ and therefore R-torsion-free. From (1.1) in Section 1.1 it is clear that $H^1(G_1, k[\mathfrak{g}])$ is isomorphic to an R-submodule of $\mathrm{Hom}_k(\mathfrak{g}, k[\mathfrak{g}])/\,\mathrm{Im}(\varphi)$, so it is also R-torsion-free. □

Let B be a Borel subgroup of G containing T, let $\mathfrak{b}$ be its Lie algebra and let $\mathfrak{u}$ be the Lie algebra of the unipotent radical U of B.

Theorem 2.2. — $H^1(B_1, k[\mathfrak{b}]) = 0$.

Proof. — Consider the restriction map $k[\mathfrak{b}]^B \to k[\mathfrak{t}]$. Under the assumptions (H1)–(H3) $\mathfrak{t}$ contains elements which are regular in $\mathfrak{g}$. Furthermore, the set of regular semisimple elements in $\mathfrak{g}$ is open in $\mathfrak{g}$. So the regular semisimple elements of $\mathfrak{g}$ in $\mathfrak{b}$ are dense in $\mathfrak{b}$. Since the union of the B-conjugates of $\mathfrak{t}$ is the set of all semisimple elements in $\mathfrak{b}$, by [3, Prop. 11.8], it is also dense in $\mathfrak{b}$. This shows that the map $k[\mathfrak{b}]^B \to k[\mathfrak{t}]$ is injective. Furthermore, $\mathrm{Ad}(g)(x) - x \in \mathfrak{u}$ for all $g \in B$ and $x \in \mathfrak{b}$ by [3, Prop. 3.17], since $DB \subseteq U$. So if we extend $f \in k[\mathfrak{t}]$ to a regular function f on $\mathfrak{b}$ by $f(x + y) = f(x)$ for all $x \in \mathfrak{t}$ and $y \in \mathfrak{u}$, then $f \in k[\mathfrak{b}]^B$. So the map

$k[\mathfrak{b}]^B \to k[\mathfrak{t}]$ is surjective, that is, restriction of functions defines an isomorphism

$$k[\mathfrak{b}]^B \xrightarrow{\sim} k[\mathfrak{t}]\,.$$

Extend a basis of $\mathfrak{t}^*$ to (linear) functions $\xi_1, \dots, \xi_n$ on $\mathfrak{b}$ in the manner indicated above. Then these functions are algebraically independent generators of $k[\mathfrak{b}]^B$, and $k[\mathfrak{b}]$ is a free $k[\mathfrak{b}]^B$-module. Clearly, the vanishing ideal of $\mathfrak{u}$ in $k[\mathfrak{b}]$ is generated by the ξ_i. Furthermore, $\min_{x\in\mathfrak{u}} \dim \mathfrak{b}_x = n$, see [12, 6.8]. We can now follow the same arguments as in the proof of Theorem 2.1. Just replace G, $\mathfrak{g}$, $\mathcal{N}$, $k[\mathfrak{g}]^G$ and the s_i by B, $\mathfrak{b}$, $\mathfrak{u}$, $k[\mathfrak{b}]^B$ and the ξ_i, and replace f_{rs} by its restriction to $\mathfrak{b}$. □

Remark 2.3. — We have $k[\mathcal{N}] = \mathrm{ind}_B^G\, k[\mathfrak{u}]$. Using [11, Lem. II.12.12a)] and the arguments from [11, II.12.2] it follows that $H^1(G_1, k[\mathcal{N}]) = \mathrm{ind}_B^G\, H^1(B_1, k[\mathfrak{u}])$. From this one can easily deduce examples with $H^1(G_1, k[\mathcal{N}]) \neq 0$.

3. The cohomology groups $H^1(G_1, k[G])$ and $H^1(B_1, k[B])$

Assume first that $G = \mathrm{GL}_n$. Put $R = k[\mathfrak{g}]^G$ and $R_1 = R[\det^{-1}]$. Then, using the fact that the $\mathfrak{g}$-action on $k[G]$ is R_1-linear, the Universal Coefficient Theorem and Theorem 2.1, we obtain

$$\begin{aligned} H^1(G_1, k[G]) = H^1\big((G_1)_{R_1}, k[G]\big) &= R_1 \otimes_R H^1\big((G_1)_R, k[\mathfrak{g}]\big) \\ &= R_1 \otimes_R H^1(G_1, k[\mathfrak{g}]) = 0\,. \end{aligned}$$

Similarly, we obtain $H^1(B_1, k[B]) = 0$.

To prove our result for the case of arbitrary reductive G we assume in this section the following:

There exists a central (see [3, 22.3]*) surjective morphism $\psi : \widetilde{G} \to G$ where $\widetilde{G}$ is a direct product of groups of the following types:*

1. *a simply connected simple algebraic group of type $\neq A$ for which p is good,*
2. SL_m *for $p \nmid m$,*
3. GL_m*,*
4. *a torus.*

THEOREM 3.1. — *$H^1(G_1, k[G]) = 0$.*

Proof. — First we reduce to the case that G is of one of the above four types. Let $\psi : \widetilde{G} \to G$ be as above. Then G is the quotient of $\widetilde{G}$ by a (schematic) central diagonalisable closed subgroup scheme $\widetilde{Z}$, see [11, II.1.18]. Let N be the image of $\widetilde{G}_1$ in G_1. Then N is normal in G_1 and G_1/N is diagonalisable. So $H^i(G_1, k[G]) = H^i(N, k[G])^{G_1/N}$, by [11, I.6.9(3)]. Furthermore, $H^i(N, k[G]) = H^i(\widetilde{G}_1, k[G])$, by [11, I.6.8(3)], since the kernel of $\widetilde{G}_1 \to N$ is central.

The group scheme $\widetilde{Z}$ also acts via the right multiplication action on $k[\widetilde{G}]$ and this action commutes with the conjugation action of $\widetilde{G}$. So $k[G] = k[\widetilde{G}]^{\widetilde{Z}}$ is a direct $\widetilde{G}$-module summand of $k[\widetilde{G}]$. So it suffices to show that $H^1(\widetilde{G}_1, k[\widetilde{G}]) = 0$. By the Künneth Theorem we may now assume that G is of one of the above four types.

For G a torus the assertion is obvious, and for $G = \mathrm{GL}_n$ we have already proved the assertion. Now assume that G is of type (1) or (2). Then G satisfies (H1)–(H3) and G is simply connected simple. By [20, 2.15] the centraliser of a semisimple group element is connected, so when the element is also regular, its centraliser is a maximal torus. As in the proof of Theorem 2.1 we are now reduced to showing that $H^1(G_1, k[G])$ has no torsion over $R := k[G]^G$.

For this it is enough that $R_\mathfrak{m} \otimes_R H^1(G_1, k[G]) = H^1(G_1, R_\mathfrak{m} \otimes_R k[G])$ has no torsion over $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. By [19, 6.11, 7.16, 8.1] the conditions of [16, Prop. 10.1] are satisfied, so $k[G]$ is a free R-module and $k[G]_\mathfrak{m} = R_\mathfrak{m} \otimes_R k[G]$ is a free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R. Furthermore, $k[G]_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} k[G]_\mathfrak{m} = k[G]/\mathfrak{m}k[G]$ is the coordinate ring of a fiber F of the adjoint quotient map. We know F is normal of codimension n, and a regular orbit closure, so $k[F]^\mathfrak{g} = k[F]^p$ by Lemma 1.1. By the local version of Lemma 1.3 the $R_\mathfrak{m}$-linear map $\varphi = f \mapsto (x \mapsto x \cdot f) : k[G]_\mathfrak{m} \to \mathrm{Hom}_k(\mathfrak{g}, k[G]_\mathfrak{m})$ we now get that $H^1(G_1, R_\mathfrak{m} \otimes_R k[G])$ has no $R_\mathfrak{m}$-torsion. □

Let B be a Borel subgroup of G.

Theorem 3.2. — *$H^1(B_1, k[B]) = 0$.*

Proof. — This follows by modifying the proof of Theorem 3.1 in the same way as the proof of Theorem 2.1 was modified to obtain the proof of Theorem 2.2. □

Remark 3.3. — One can also prove Theorem 3.2 assuming (H1)–(H3). The point is that it is obvious that restriction of functions always defines an isomorphism $k[B]^B \xrightarrow{\sim} k[T]$.

Remark 3.4. — We briefly discuss the B-cohomology of $k[B]$ and $k[\mathfrak{b}]$. From [13, Thm. 1.13, Thm. 1.7(a)(ii)] it is immediate that $H^i(B, k[B]) = 0$ for all $i > 0$. Now assume that there exists a central surjective morphism

$\psi : \widetilde{G} \to G$ where $\widetilde{G}$ is a direct product of groups of the types (1)–(4) mentioned before, except that for type (2) we drop the condition on p. Then we deduce from the arguments from the proof of [1, Prop. 4.4] that $H^i(B, k[\mathfrak{b}]) = 0$ for all $i > 0$ as follows. First we reduce as in the proof of Theorem 3.1 to the case that G is simple of type (1) or (2) and then we deal with type (2) as in [1]. Now assume G is of type (1) and let I be the vanishing ideal of B in $k[G]$. As in [1] write

$$\mathfrak{m} = M \oplus \mathfrak{m}^2 \tag{3.1}$$

where $\mathfrak{m}$ is the vanishing ideal in $k[G]$ of the unit element of G and $M \cong \mathfrak{g}^*$ as G-modules. It suffices to show that $I = I \cap M + I \cap \mathfrak{m}^2$, since then we get a decomposition analogous to (3.1) for $k[B]$ and we can finish as in [1]. Let $f \in I$. Then the M-component of f correspond to $df \in \mathfrak{g}^*$ which vanishes on $\mathfrak{b}$. This means it corresponds under the trace form of the chosen representation $\rho : G \to V$ (the adjoint representation for exceptional types) to an element $x \in \mathfrak{u}$. So the M-component of f is $g \mapsto \mathrm{tr}\big(\rho(g)d\rho(x)\big)$ which vanishes on B. But then the $\mathfrak{m}^2$-component of f must also vanish on B.

4. The cohomology groups for the higher Frobenius kernels

In this section we will generalise the results from the previous two sections to all Frobenius kernels G_r, $r \geqslant 1$.

Lemma 4.1. — *Let G be a linear algebraic group over k acting on a normal affine variety X over k. If $\max_{x\in X} \mathrm{codim}_{\mathfrak{g}}\, \mathfrak{g}_x = \dim X$, then $k[X]^{G_r} = k[X]^{p^r}$ for all integers $r \geqslant 1$.*

Proof. — Since $\mathrm{codim}_{\mathfrak{g}}\, \mathfrak{g}_x \leqslant \mathrm{codim}_G\, G_x \leqslant \dim(X)$ and $\max_{x\in X} \mathrm{codim}_{\mathfrak{g}}\, \mathfrak{g}_x = \dim X$ we must have that for $x \in X$ with $\mathrm{codim}_{\mathfrak{g}}\, \mathfrak{g}_x = \dim X$ the schematic centraliser of x in G is reduced. So $(G_r)_x = (G_x)_r$ and

$$\begin{aligned}(G_r : (G_r)_x) &:= \dim(k[G_r])/\dim(k[(G_r)_x])\\ &= p^{r\dim(G)}/p^{r\dim(G_x)} = p^{r\dim(X)}.\end{aligned}$$

By [17, Thm. 2.1 (5)] we get $[k(X) : k(X)^{G_r}] = p^{r\dim(X)}$. By [4, Cor. 3 to Thm. V.16.6.4] and the tower law we have $[k(X) : k(X)^{p^r}] = p^{r\dim(X)}$. So $k(X)^{G_r} = k(X)^{p^r}$, since we always have $\supseteq$. Clearly, $k(X)^{p^r} = \mathrm{Frac}(k[X]^{p^r})$, $k(X)^{G_r} = \mathrm{Frac}(k[X]^{G_r})$ and $k[X]^{G_r}$ is integral over $k[X]^{p^r}$. Since X is normal variety, $k[X]^{p^r} \cong k[X]$ is a normal ring. It follows that $k[X]^{G_r} = k[X]^{p^r}$. ☐

Theorem 4.2. — *Let r be an integer $\geqslant 1$.*

(1) *Under the assumptions of Section 2 we have*

$$H^1(G_r, k[\mathfrak{g}]) = 0 \text{ and } H^1(B_r, k[\mathfrak{b}]) = 0.$$

(2) *Under the assumptions of Section 3 we have*

$$H^1(G_r, k[G]) = 0 \text{ and } H^1(B_r, k[B]) = 0.$$

Proof.

(1). — Let (H, M) be the group and module in question, i.e. $(G, k[\mathfrak{g}])$ or $(B, k[\mathfrak{b}])$. Put $R = k[\mathfrak{h}]^H$. Let φ be the first map in the Hochschild complex of the H_r-module M, see [11, I.4.14]:

$$\varphi = f \mapsto (\Delta_M(f) - 1 \otimes f) : M \to k[H_r] \otimes M.$$

Then the induced map $\overline{\varphi} : M/R^+M \to k[H_r] \otimes (M/R^+M)$ is the first map in the Hochschild complex of the H_r-module M/R^+M which is $k[\mathcal{N}]$ or $k[\mathfrak{u}]$. So $\mathrm{Ker}(\varphi) = M^{H_r}$ and $\mathrm{Ker}(\overline{\varphi}) = (M/R^+M)^{H_r}$. Now the proof is the same as that of the corresponding result in Section 2, except that we work with the above map φ and instead of Lemma 1.1 we apply Lemma 4.1.

(2). — Let (H, M) be the group and module in question, i.e. $(G, k[G])$ or $(B, k[B])$. As in the proof of the corresponding result in Section 3 we reduce to the case that G is simple of type (1) or (2). Put $R = k[H]^H$. Fix a maximal ideal $\mathfrak{m}$ of R. Let φ be the first map in the Hochschild complex of the H_r-module $M_{\mathfrak{m}}$. Then the induced map $\overline{\varphi} : M_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}M_{\mathfrak{m}} \to k[H_r] \otimes M_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}M_{\mathfrak{m}}$ is the first map in the Hochschild complex of the H_r-module $M_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}M_{\mathfrak{m}} = M/\mathfrak{m}M$ which is the coordinate ring of the fiber of $H \to H/\!/H$ over the point $\mathfrak{m}$. So $\mathrm{Ker}(\varphi) = (M_{\mathfrak{m}})^{H_r}$ and $\mathrm{Ker}(\overline{\varphi}) = (M/\mathfrak{m}M)^{H_r}$. Now the proof is the same as that of the corresponding result in Section 3, except that we work with the above map φ and instead of Lemma 1.1 we apply Lemma 4.1. □

Remark 4.3. — For G classical with natural module $V = k^n$ we consider the cohomology groups $H^1(G_r, S^iV)$ and $H^1(G_r, S^i(V^*))$.

Results about these modules can mostly easily be deduced from results on induced modules in the literature. For induced modules one can reduce to B_r-cohomology using the following result of Andersen–Jantzen for general G. Let B be a Borel subgroup of G with unipotent radical U and let T be a maximal torus of B. For $\lambda \in X(T)$, the character group of T, we denote by $\nabla(\lambda)$, the G-module induced from the 1-dimensional B-module given by λ. We call the roots of T in the opposite Borel subgroup B^+ positive. By [11, II.12.2] we have for λ dominant

$$(4.1)\qquad H^1(G_r,\nabla(\lambda))^{[-r]}\cong \mathrm{ind}_B^G(H^1(B_r,\lambda)^{[-r]}).$$

Below we will always take $\lambda=\varpi_1$ the first non-constant diagonal matrix coordinate. First take $G=\mathrm{GL}_n$. Let B and T be the lower triangular matrices and the diagonal matrices. Then the character group $X(T)$ of T identifies with $\mathbb{Z}^n$. Let ε_1 be the first standard basis element of $X(T)$, i.e. the character $D\mapsto D_{ii}$. Then $S^iV=\nabla(i\varepsilon_1)$ and $S^i(V^*)=\nabla(-i\varepsilon_n)$. Replacing $\mathfrak{u}^{*[s]}$ by $\lambda\otimes\mathfrak{u}^{*[s]}$ for $\lambda=i\varepsilon_1$ or $\lambda=-i\varepsilon_n$ in the proof of [11, Lem. II.12.1] and using (4.1) we obtain $H^1(G_r,S^iV)=H^1(G_r,S^i(V^*))=0$.

Now take $G=\mathrm{SL}_n$. Then $S^iV=\nabla(i\varpi_1)$ and $S^i(V^*)=\nabla(i\varpi_{n-1})$, where ϖ_j denotes the j-th fundamental dominant weight. From [2, Cor. 3.2(a)] we easily deduce that $H^1(G_r,S^iV)\neq 0$ if and only if $H^1(G_r,S^i(V^*))\neq 0$ if and only if $n=2$ and $p^r\mid i+2p^s$ for some $s\in\{0,\dots,r-1\}$, or $n=3$, $p=2$ and $2^r\mid i-2^{r-1}$.

For $G=\mathrm{Sp}_n$, $n\geqslant 4$ even, we deduce using $S^i(V)=\nabla(i\varpi_1)$ and [2, Cor. 3.2(a)] that $H^1(G_r,S^iV)\neq 0$ if and only if $p=2$ and i is odd.

Now let G be the special orthogonal group SO_n, $n\geqslant 4$, as defined in [18, Ex. 7.4.7(3), (4), (6), (7)] (when $p=2$ this is an abuse of notation). Note that $V\cong V^*$ unless n is odd and $p=2$. Although the simply connected cover $\widetilde{G}\to G$ need not be separable, it still follows from [11, I.6.8(3), I.6.9(3)] that $H^1(G_r,M)=H^1(\widetilde{G}_r,M)^{T_r}$ for any G-module M, and $H^1(B_r,M)=H^1(\widetilde{B}_r,M)^{T_r}$ for any B-module M. So one has to pick out the weight spaces of the weights in $p^rX(T)\subseteq p^rX(\widetilde{T})$. For $n\geqslant 8$ it follows from [2, Cor. 3.2(a)] that $H^1(\widetilde{G}_r,\nabla(i\varpi_1))=0$ for all $i\geqslant 0$. For general $n\geqslant 4$ we proceed as follows. From [2, Sect. 2.5–2.7] we deduce that all weights of $H^1(B_r,i\varpi_1)$ are of the form $i\varpi_1+p^s\alpha$ for some $s\in\{0,\dots,r-1\}$ and some α simple or "long" (i.e. there is a shorter root). Since such weights don't occur in $p^rX(T)$ for SO_n, $n\geqslant 4$, we get that $H^1(B_r,i\varpi_1)=0$, and therefore by (4.1) $H^1(G_r,\nabla(i\varpi_1))=0$ for all $i\geqslant 0$. By [11, II.2.17,18] $S^i(V^*)$ has a filtration with sections $\nabla(i\varpi_1),\nabla((i-2)\varpi_1),\dots$. So $H^1(G_r,S^i(V^*))=0$ for all $i\geqslant 0$.

The fact that the weights of $H^1(B_r,i\varpi_1)$ have the form stated above can been seen more directly as follows. First one observes that 1-cocyles in the Hochschild complex of a U_r-module M can be seen as the linear maps $D:\mathrm{Dist}^+(U_r)\to M$ with $D(ab)=aD(b)$ for all $a\in\mathrm{Dist}(U_r)$ and $b\in\mathrm{Dist}^+(U_r)$. Here $\mathrm{Dist}^+(U_r)$ denotes the distributions without constant term, i.e. the distributions a with $a(1)=0$. Then one shows that, outside type G_2, $\mathrm{Dist}(U_r)$ is generated by the $\mathrm{Dist}(U_{-\alpha,r})$ with α simple or long.(2)

(2) If p is not special in the sense of [9], then (also in type G_2) $\mathrm{Dist}(U_r)$ is generated by the $\mathrm{Dist}(U_{-\alpha,r})$ with α simple.

It follows that $H^1(U_r, M)$ is a subquotient of $M \otimes \bigoplus_{\alpha,\, 0\leqslant s<r} \mathfrak{u}_{-\alpha}^{*[s]}$, the α simple or long. Now use that, for M a B_r-module, $H^1(B_r, M) = H^1(U_r, M)^{T_r}$.

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